

On self-complementary chordal graphs

by

K. Balaji

TH
MATH/1997/P
B182e



DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Kanpur

DECEMBER, 1997

On self-complementary chordal graphs

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

K.Balaji

to the

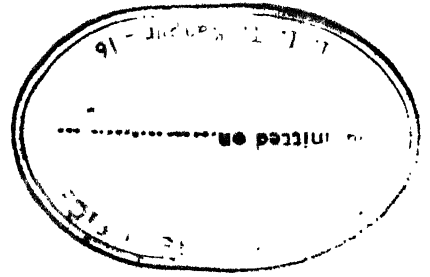
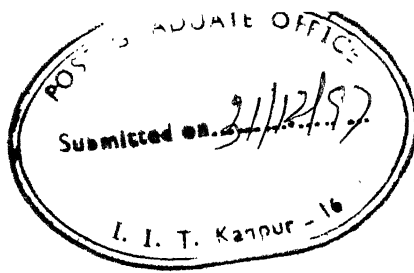
Department of Mathematics
Indian Institute of Technology, Kanpur
December, 1997

174 JUN 2000 / MATH
CENTRAL LIBRARY
I. I. T., KANPUR
131082

TH
MATH/1997/P
B182.6

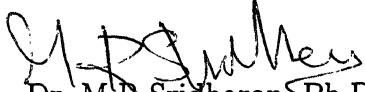


A131082



Certificate

It is certified that the work contained in the thesis entitled "*On self-complementary chordal graphs*" by *K. Balaji* has been carried out under my supervision and guidance and that this work has not been submitted elsewhere for any degree or diploma.


Dr. M.R. Sridharan, Ph.D.,

Professor,

Department of Mathematics,
Indian Institute of Technology
Kanpur.

Acknowledgement

First of all, I would like to record my sincere thanks to my thesis supervisor Prof.M.R.Sridharan who is a constant source of inspiration throughout my stay at IIT, Kanpur. I am indebted to him for introducing me to this topic and making my most cherished dream of getting a Ph.D. degree, come true. I owe him a lot and I am ever grateful to him for all that he has done for me. He is a friend, philosopher and guide. I also thank Mrs.Sridharan and Master.Vallabh for the hospitality they provided whenever I visited them.

I would like to express my sincere thanks to all my teachers who kindled my interest in Mathematics and helped me to understand its philosophy. Special thanks are due to Dr.B.V.Ratish Kumar for his kind help and discussions on our overlapping research interests. I would like to thank Prof.Punyatma Singh, Prof.S.K.Gupta and other faculty members of the Department of Mathematics, IIT, Kanpur, who had provided me all the facilities in the Department for carrying out my Ph.D. research.

I would also like to thank P.V.S.N.Murthy, Manish, Mahesh, S.B.Rao, Guru Prem Prasad, P.Muthu and other Department colleagues who had been very friendly to me throughout my stay at IIT, Kanpur. Thanks are also due to Mr.Jain, Mr.Srivatsava and other non-teaching staff of the Department of Mathematics and the DOAA Office, IIT, Kanpur, for maintaining my database throughout my stay here.

I take this opportunity to express my sincere thanks to my friends who had helped me at several critical junctures. Special thanks are due to my friends R.Kalyanaraman, K.Sivakumar, S.Sivagurunathan, S.Rajesh, Ravi, N.Babu, V.Venkatrathnam, V.Anand, S.Balasubramanian, Gomathi Shankar, R.Joye, Sridevi and Viji. They made my stay at IIT Kanpur very enjoyable. I can never forget the great hospitality provided to me by Major.Parthasarathy, Mrs.Parthasarathy and Bhargavi.

It goes without saying that my parents, my sisters and my relatives had been a source of great support and encouragement in all my endeavours.

Last but not the least, I would like to thank the Indian Institute of Technology, Kanpur, for providing all the facilities throughout my stay here.

K. Balaji
21.12.87
K. Balaji

Dedicated in memory
of my father
Shri. P.S.V.Kannan

Synopsis

Perfect Graphs and several classes of perfect graphs had been studied because of the nice combinatorial structure and interesting applications [24], [101] and [179].

The four problems, namely,

- (i) finding the clique number of a graph
- (ii) finding the chromatic number of a graph
- (iii) finding the stability number of a graph
- (iv) finding the clique covering number of a graph

which are NP-hard in general [91] can be solved in polynomial time when restricted to perfect graphs [105]. This result has intensified the algorithmic interest in perfect graphs. In 1962 Berge [14] posed the following conjecture (see also [24] and [101]).

Conjecture 1 : *For a graph G the following conditions are equivalent.*

- (i) G is α -perfect.
- (ii) G is χ -perfect.
- (iii) G has no induced subgraph isomorphic to C_{2k+1} or \bar{C}_{2k+1} for $k \geq 2$.

Lovasz [133] established that a graph is χ -perfect if and only if it is α -perfect. This is called the Perfect Graph Theorem. As a consequence of the Perfect Graph Theorem, Conjecture 1 reduces to Conjecture 2.

Conjecture 2 : *A graph is a perfect graph (χ -perfect or α -perfect) if and only if it has no induced subgraph isomorphic to C_{2k+1} or \bar{C}_{2k+1} for all $k \geq 2$.*

Conjecture 2 is called the Strong Perfect Graph Conjecture (SPGC) and it is still unsettled. A class of graphs is said to be complete for a conjecture if the truth of the conjecture on this class implies the truth of the conjecture in general. Corneil [70] has identified self-complementary graphs, regular graphs and various other classes of graphs to

be complete classes for SPGC. This result of Corneil motivates the study of the various classes of self-complementary perfect graphs. In this thesis we study the self-complementary chordal graphs, a class of self-complementary perfect graphs.

In Chapter 1 entitled ‘Introduction’ we discuss briefly about the various results on s.c. graphs and chordal graphs available in the literature.

In Chapter 2 entitled ‘On the characterisations for self-complementary graphs to be chordal’ we obtain the following characterisations.

- (i) Let G be a self-complementary (s.c.) graph. Then G is chordal if and only if $\omega(G) = 2n$ when $p = 4n$ and $\omega(G) = 2n + 1$ when $p = 4n + 1$ where p denotes the number of vertices of G and $\omega(G)$ denotes the clique number of G .
- (ii) Let G be a s.c. graph. Then G is chordal if and only if $\alpha(G) = 2n$ when $p = 4n$ and $\alpha(G) = 2n + 1$ when $p = 4n + 1$ where $\alpha(G)$ denotes the stability number of G .
- (iii) Let G be a s.c. graph. Then G is chordal if and only if G has no induced subgraph isomorphic to C_4 when $p = 4n$ and G has no induced subgraph isomorphic to C_4 or C_5 when $p = 4n + 1$ where C_n denotes the cycle with n vertices.
- (iv) Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then G is chordal if and only if $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ when $p = 4n$ and $\sum_{i=1}^{2n} d_i = 6n^2$ when $p = 4n + 1$.

The results (i) and (ii) have been published in reputed journals [204] and [205].

In Chapter 3 entitled ‘On the existence and the construction of self-complementary chordal graphs’ we prove that s.c. chordal graphs with p vertices exist if and only if $p = 4n$ or $p = 4n + 1$ for some positive integer n . We also give algorithms to construct all s.c. chordal graphs with $4n$ and $4n + 1$ vertices for all positive integers n . Though these algorithms construct all s.c. chordal graphs with $4n$ and $4n + 1$, vertices they may construct the same graph (with different labellings) repeatedly. So the isomorphism problem of s.c. chordal graphs plays a major role in the catalogue compilation of s.c. chordal graphs (list of non-isomorphic s.c. chordal graphs).

In Chapter 4 entitled ‘On the isomorphism and the catalogue compilation of self-complementary chordal graphs’ we prove that the isomorphism of s.c. chordal graphs, the

isomorphism of s.c. chordal graphs with $4n$ vertices and the isomorphism of s.c. chordal graphs with $4n+1$ vertices are polynomially equivalent. Alter [5], Faradzhev [78], Kropar et. al. [127], Morris [143] and Venkatachalam [225] had compiled the catalogue of s.c. graphs with a few vertices (upto 12 vertices). From these we recognise those graphs which are chordal by using the recognition algorithms given in this chapter and obtain the catalogue of s.c. chordal graphs with atmost 12 vertices. We also obtain the catalogue of s.c. chordal graphs with 13 vertices by giving a method for obtaining all non-isomorphic s.c. chordal graphs with $4n+1$ vertices from the set of all non-isomorphic s.c. chordal graphs with $4n$ vertices.

The results in Chapters 3 and 4 have been submitted for publication [206] and [207].

Graph parameters had been studied because of the usefulness in determining the structure of the graph. In Chapter 5 entitled 'Self-complementary chordal graphs and some graph parameters' we study the following graph parameters for s.c. chordal graphs.

- (a) Chromatic number.
- (b) Chromatic index.
- (c) Domination number.
- (d) Spectrum.

In Section 5.2 of Chapter 5 we prove that the chromatic number of s.c. chordal graphs with $4n$ vertices is $2n$ and that of s.c. chordal graphs with $4n+1$ vertices is $2n+1$. We also get bounds on the chromatic number of s.c. perfect graphs. We prove that the upper bounds are attained if and only if the graph is s.c. chordal.

In Section 5.3 of Chapter 5 we give a sufficient condition for a s.c. chordal graph with $4n$ vertices to be a Class 1 graph (a graph whose chromatic index is equal to its maximum degree) and establish the Class 1 property for some classes of s.c. chordal graphs with $4n$ vertices. We also obtain bounds for the chromatic index of s.c. chordal graphs.

In Section 5.4 of Chapter 5 we prove that the domination number and the independent domination number can be atmost $2n$ for s.c. chordal graphs with $4n$ vertices and atmost $2n+1$ for s.c. chordal graphs with $4n+1$ vertices.

In Section 5.5 of Chapter 5 we prove that the least positive integer for which there exist cospectral s.c. chordal graphs is 12. We also obtain bounds for the maximum eigen-value of a s.c. chordal graph.

In Chapter 6 entitled 'Conclusion' we offer some research problems on s.c. chordal graphs.

List of Publications

- (i) M.R.Sridharan and K.Balaji, 'On self-complementary chordal graphs', National Academy Science Letters 20 (1997).
- (ii) M.R.Sridharan and K.Balaji, 'Characterisation of self-complementary chordal graphs', accepted for publication in Discrete Mathematics.
- (iii) M.R.Sridharan and K.Balaji, 'On construction of self-complementary chordal graphs', submitted for publication.
- (iv) M.R.Sridharan and K.Balaji, 'On isomorphism of self-complementary chordal graphs', submitted for publication.

Contents

1	Introduction	1
1.1	General introduction	1
1.2	Outline of the thesis	6
1.3	Basic definitions and notations	8
2	On the characterisations for self-complementary graphs to be chordal	18
2.1	Introduction	18
2.2	Basic Results	19
2.3	A characterisation for self-complementary graphs to be chordal in terms of the clique number	20
2.4	A characterisation for self-complementary graphs to be chordal in terms of the stability number	30
2.5	A characterisation for self-complementary graphs to be chordal in terms of the induced cycles	31
2.6	A characterisation for self-complementary graphs to be chordal in terms of the degree sequence	38
3	On the existence and the construction of self-complementary chordal graphs	40
3.1	Introduction	40
3.2	Algorithms for constructing self-complementary graphs	40
3.3	On the existence of self-complementary chordal graphs	44
3.4	Algorithms for constructing self-complementary chordal graphs	46

4	On the isomorphism and the catalogue compilation of self-complementary chordal graphs	67
4.1	Introduction	67
4.2	Algorithmic complexity of the isomorphism of self-complementary chordal graphs	68
4.3	On the catalogue compilation of self-complementary chordal graphs	75
5	Self-complementary chordal graphs and some graph parameters	108
5.1	Introduction	108
5.2	On the chromatic number of self-complementary chordal graphs	109
5.3	On the chromatic index of self-complementary chordal graphs	111
5.4	On the domination number of self-complementary chordal graphs	114
5.5	On the spectrum of self-complementary chordal graphs	116
6	Conclusion	125

List of Figures

Figure 1.1(a)	16
Figure 1.1(b)	16
Figure 1.2	17
Figure 1.3	17
Figure 3.1(a)	55
Figure 3.1(b)	55
Figure 3.2(a)	56
Figure 3.2(b)	56
Figure 3.3(a)	57
Figure 3.3(b)	57
Figure 3.3(c)	58
Figure 3.4(a)	59
Figure 3.4(b)	59
Figure 3.4(c)	60
Figure 3.5(a)	61
Figure 3.5(b)	61
Figure 3.6(a)	62
Figure 3.6(b)	62
Figure 3.7(a)	63
Figure 3.7(b)	63
Figure 3.8(a)	64
Figure 3.8(b)	64
Figure 3.9(a)	65
Figure 3.9(b)	65
Figure 3.10 (a)	66
Figure 3.10 (b)	66
Figure 4.1 (a)	100
Figure 4.1 (b)	101

Figure 4.2 (a)	102
Figure 4.2 (b)	103
Figure 4.3 (a)	104
Figure 4.3 (b)	105
Figure 4.4 (a)	106
Figure 4.4 (b)	107
Figure 5.1	123
Figure 5.2	124

Chapter 1

Introduction

1.1 General introduction

Perfect Graphs and several classes of perfect graphs had been studied because of the nice combinatorial structure and interesting applications [24], [52], [77], [101], [102], [179], [220] and [221].

The four problems, namely,

- (i) finding the clique number of a graph
- (ii) finding the chromatic number of a graph
- (iii) finding the stability number of a graph
- (iv) finding the clique covering number of a graph

which are NP-hard in general [91] can be solved in polynomial time when restricted to perfect graphs [105]. This result has intensified the algorithmic interest in perfect graphs. Soon after the introduction of perfect graphs many started to identify these graphs. Berge [16] showed that many familiar classes of graphs such as chordal graphs (also known as triangulated graphs in the literature), comparability graphs, interval graphs, unimodular graphs and line graphs of bipartite graphs are perfect. Foldes and Hammer [85] established that split graphs belong to the class of chordal graphs. Thus these graphs are also perfect. The idea of splittance of a graph namely, the minimum number of edges to be added or deleted in

order to produce a split graph was introduced by Hammer and Simone [109]. According to this definition split graphs are those graphs whose splittance is zero. Interval graphs (a class of perfect graphs) became popular because of its applications in traffic light phasing [179], seriation [101] (that is an attempt to place a set of items in their proper chronological order) and certain other optimization problems [101] and [179]. Hajos [107] first posed the problem of characterising interval graphs. The famed molecular biologist Benzer [11] was also trying to find the answer to a related problem in his investigations of the fine structure of the gene. The first characterisation of interval graphs appeared in 1962 by Lekkerker and Boland [130] followed by Gilmore and Hoffmann [99]. Fulkerson and Gross [89] gave still another characterisation. Interval graphs are perfect because they are chordal [99]. Closely related to interval graphs are circular-arc graphs. Every interval graph is a circular-arc graph but the converse is not true. Circular-arc graphs are not perfect in general. For example C_{2k+1} for all $k \geq 2$ belong to this class and are not perfect. We refer to [120], [137], [210], [211] and [223] for applications of these graphs. Pnueli et. al. [155] proved that a graph G is a permutation graph if and only if G and \bar{G} are comparability graphs. This implies that permutation graphs are perfect. Isomorphism of permutation graphs has been studied in [64]. For other results on perfect graphs and their applications we refer to [13], [15], [17], [18], [19], [20], [21], [22], [23], [25], [29], [48], [49], [50], [53], [198], [230], [100], [101], [110], [118], [119], [132], [135], [138], [148], [157], [169], [170], [176], [179], [194], [195], [212], [219] and [224].

Motivated by Shannon's work [195] in 1962 Berge [14] posed the following conjecture (see also [24] and [101]).

Conjecture 1.1 : *For a graph G the following conditions are equivalent.*

- (i) G is α -perfect.
- (ii) G is χ -perfect.
- (iii) G has no induced subgraph isomorphic to C_{2k+1} or \bar{C}_{2k+1} for $k \geq 2$.

During the initial stages the concepts of α -perfectness and χ -perfectness were thought to be distinct and a perfect graph was defined to be a graph which is both χ -perfect and α -

perfect. Lovasz [133] established that a graph is χ -perfect if and only if it is α -perfect. This is called the Perfect Graph Theorem. Lovasz in [132] gave another equivalent condition for α -perfectness and χ -perfectness. Golumbic in [101] observed that the Perfect Graph Theorem had been almost proved by Fulkerson. As a consequence of the Perfect Graph Theorem, Conjecture1.1 reduces to Conjecture 1.2.

Conjecture 1.2 : *A graph is a perfect graph (χ -perfect or α -perfect) if and only if it has no induced subgraph isomorphic to C_{2k+1} or \bar{C}_{2k+1} for all $k \geq 2$.*

Conjecture1.2 is called the Strong Perfect Graph Conjecture (SPGC) and it is still unsettled. A class of graphs is said to be valid for a conjecture if the conjecture is true for this restricted class of graphs. Many special classes of graphs called $K_{1,3}$ -free graphs [152], toroidal graphs [104], $(K_4 - e)$ -free graphs [153], planar graphs [221] and many other classes of graphs [42], [230], [129], [136], [146] and [222] were shown to be valid classes for SPGC. A class of graphs is said to be complete for a conjecture if the truth of the conjecture on this class implies the truth of the conjecture in general. Corneil [70] has identified self-complementary graphs, regular graphs and various other classes of graphs to be complete classes for SPGC. This result of Corneil motivates the study of various classes of self-complementary perfect graphs and self-complementary imperfect graphs. In this thesis we study the self-complementary chordal graphs, a class of self-complementary perfect graphs. For various other results on SPGC we refer to [37], [139], [147], [170], [208] and [220].

The class of self-complementary chordal graphs enjoy both the properties of self-complementary graphs and the properties of chordal graphs. The class of self-complementary graphs and the class of chordal graphs had been widely studied in the literature. We discuss briefly about the various results available in these two classes.

(i) **Self-complementary graphs (s.c. graphs) :** The existence problem of s.c. graphs was solved independently by Ringel [178] and Sachs [190]. They prove that s.c. graphs with p vertices exist if and only if $p=4n$ or $p=4n+1$ for some positive integer n . Ringel [178] obtained algorithms to construct s.c. graphs with $4n$ and $4n+1$ vertices. Algorithms for constructing s.c. graphs with $4n$ and $4n+1$ vertices were also obtained by Gibbs [98] by

modifying Ringel's algorithms. Harary [111] posed the problem of counting non-isomorphic s.c. graphs given the number of vertices. A complete solution for this problem was given by Read [171] and [172]. His method is based on enumeration theory originated by Redfield [175] and Polya [156] developed further by DeBruijn [35] and [36] and Harary et. al. [113]. The number of s.c. graphs (non-isomorphic) s_p for a given number of vertices p is given in the following table for $p \leq 17$.

p	1	4	5	8	9	12	13	16	17
s_p	1	1	2	10	36	720	5600	703760	11220000

An asymptotic formula for s_p as $p \rightarrow \infty$ was derived by Palmer [149]; see also Robinson [180], Sridharan [203] and Schwenk [192]. Further results concerning enumeration of s.c. graphs can be found in [3], [115], [149], [154], [172], [173], [202], [203] and [209]. The problem of deciding whether two given graphs are isomorphic or not is called the isomorphism problem. Isomorphism of s.c. graphs and regular s.c. graphs was discussed by Colbourn et. al. [65] and [66]. They prove that the isomorphism of these classes is polynomially equivalent to the general graph isomorphism. The problem of deciding whether a given graph is s.c. or not is called the recognition problem of s.c. graphs. Colbourn et. al. [65] prove that the recognition of s.c. graphs is polynomially equivalent to the general graph isomorphism. Catalogue of s.c. graphs with small number of vertices was compiled by Alter [5], Faradzhev [78], Kropar et. al. [127], Morris [143] and Venkatachalam [225]. Clapham and Kleitman [61] had obtained a necessary and sufficient condition for a degree sequence to be the degree sequence of a s.c. graph. Further results on the degree sequence of s.c. graphs are obtained in [46], [57], [73] and [74]. The existence of paths or cycles of prescribed length in s.c. graphs was investigated by Camion [41], Clapham [55] [56] and [58] and Rao [164] [165] [166] and [167]. In Camion [41] and Clapham [55] and [58] it is proved that each s.c. graph has a Hamiltonian path. S.c. graphs with atmost 5 vertices have an odd number of Hamiltonian paths and s.c. graphs with atleast 6 vertices have an even number of Hamiltonian paths [41] and [166]. According to Rao [163] each s.c. graph with p vertices $p > 5$ has a cycle of length i for each $2 \leq i \leq p-2$ but need not have a cycle of length $p-1$ or p . However if it does have

a Hamiltonian cycle then it contains cycles of all lengths $2 \leq i \leq p$ (pancyclic) [163]. The problem of the existence of Hamiltonian cycle in s.c. graphs was solved by Rao [165]. Results on the number of triangles in s.c. graphs was given by Clapham [54], Radhakrishnan Nair et. al. [160] and Rao [167]. Chao et. al. [45] studied the s.c. graphs with a given chromatic number. They prove that for each n the maximum number of vertices of any n – chromatic s.c. graph is n^2 . Further, for $n \geq 3$ they constructed n -chromatic s.c. graphs of diameter 2 and 3. Shannon capacity of s.c. graphs was investigated by Lovasz [134]. Various other results on s.c. graphs can be found in [2], [6], [8], [34], [51], [47], [56], [59], [60], [71], [88], [114], [140], [161], [162], [168], [187], [188], [189], [191], [213], [235] and [236].

(ii) **Chordal graphs** : Chordal graphs are also known as triangulated graphs, rigid circuit graphs, monotone transitive graphs and perfect elimination graphs in the literature. This class of perfect graphs has been extensively studied in the literature [12], [24], [39], [43], [75], [76], [80], [93], [94], [97], [101], [102], [106], [121], [177], [179], [181], [182], [183], [196], [199], [200] and [218]. Chordal graphs were introduced by Hajnal and Suranyi [106]. They prove that these graphs are α -perfect. Berge [12] proved the χ -perfectness of these graphs. These graphs find applications in evolutionary trees [39], archaeology [38], facility location [44], scheduling [150] and solutions of sparse systems of linear equations [182]. Many graph problems including the four classical optimization problems that are NP-hard for general graphs can be solved in polynomial time in chordal graphs [91] though testing Hamiltonicity [67], determining the domination number [31] and other problems [91] and [123] are NP-Complete for this class too. A concept called Perfect Elimination Ordering (PEO) is important in chordal graphs. It turns out that all the existing chordal graph recognition algorithms and many optimization problems including the four classical ones in chordal graphs make use of PEO [91], [101], [131], [181], [184], [197], [215] and [216]. Fulkerson and Gross [89] characterised these graphs as the graphs having a perfect elimination scheme. Dirac [75] shows these graphs as the graphs for which every minimal vertex separator induces a complete subgraph. Buneman [39], Gavril [95] and Walter [229] prove that a graph is chordal if and only if it is an intersection graph of the subtrees of a tree. Various other characterisations of chordal graphs are given in [10], [24] and [101]. Various subclasses of chordal graphs namely, split graphs, strongly chordal graphs, threshold graphs, interval graphs and k -trees had been

studied in [7], [9], [11], [26], [27], [30], [32], [40], [50], [80], [84] [85], [86], [87], [89], [99], [102], [109], [110], [126], [130], [158], [159] [183], [201] and [231]. Chordal graphs are discussed in [96], [124], [128], [142], [144], [185], [186], [214], [218], [228], [229], [233] and [234].

1.2 Outline of the thesis

In this thesis we study the class of s.c. chordal graphs which is a subclass of the class of s.c. perfect graphs.

In Chapter 2 entitled ‘On the characterisations for self-complementary graphs to be chordal’ we obtain the following characterisations.

- (i) Let G be a s.c. graph. Then G is chordal if and only if $\omega(G) = 2n$ when $p = 4n$ and $\omega(G) = 2n + 1$ when $p = 4n + 1$ where p denotes the number of vertices of G and $\omega(G)$ denotes the clique number of G .
- (ii) Let G be a s.c. graph. Then G is chordal if and only if $\alpha(G) = 2n$ when $p = 4n$ and $\alpha(G) = 2n + 1$ when $p = 4n + 1$ where $\alpha(G)$ denotes the stability number of G .
- (iii) Let G be a s.c. graph. Then G is chordal if and only if G has no induced subgraph isomorphic to C_4 when $p = 4n$ and G has no induced subgraph isomorphic to C_4 or C_5 when $p = 4n + 1$ where C_n denotes the cycle with n vertices.
- (iv) Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then G is chordal if and only if $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ when $p = 4n$ and $\sum_{i=1}^{2n} d_i = 6n^2$ when $p = 4n + 1$.

Characterisations (i) and (ii) had been published in reputed journals [204] and [205].

In Chapter 3 entitled ‘On the existence and the construction of self-complementary chordal graphs’ we prove that s.c. chordal graphs with p vertices exist if and only if $p = 4n$ or $p = 4n + 1$ for some positive integer n . We also give algorithms to construct all s.c. chordal graphs with $4n$ and $4n + 1$ vertices for any positive integer n . Though these algorithms construct all s.c. chordal graphs with $4n$ and $4n + 1$ vertices, they may construct the same graph (with different labellings) repetitively. So the isomorphism problem of s.c. chordal

graphs plays a major role in the catalogue compilation of s.c. chordal graphs (list of non-isomorphic s.c. chordal graphs).

Isomorphism problem is still unsolved inspite of the several attempts made by many researchers to solve it [33] [63] [65] [66] [68] [69] [92] [123] and [174]. Isomorphism of s.c. chordal graphs apart from playing an important role in the catalogue compilation of s.c. chordal graphs seems to have close relationship with the solvability of the isomorphism problem. It appears that the isomorphism problem is solvable if and only if it is when restricted to s.c. chordal graphs. In Chapter 4 entitled 'On the isomorphism and the catalogue compilation of self-complementary chordal graphs' we prove that the isomorphism of s.c. chordal graphs, the isomorphism of s.c. chordal graphs with $4n$ vertices and the isomorphism of s.c. chordal graphs with $4n+1$ vertices are polynomially equivalent. Alter [5], Faradzhev [78], Kropar et al. [127], Morris [143] and Venkatachalam [225] had compiled a catalogue of s.c. graphs with a few vertices (upto 12 vertices). From this catalogue we recognise those graphs which are chordal by using the recognition algorithms given in this chapter and obtain the catalogue of s.c. chordal graphs with atmost 12 vertices. We also obtain the catalogue of s.c. chordal graphs with 13 vertices by giving a method for obtaining all non-isomorphic s.c. chordal graphs with $4n+1$ vertices from the set of all non-isomorphic s.c. chordal graphs with $4n$ vertices.

The results in Chapters 3 and 4 have been submitted for publication [206] and [207].

In Chapter 5 entitled 'Self-complementary chordal graphs and some graph parameters' we study the following graph parameters for s.c. chordal graphs.

- (a) Chromatic number.
- (b) Chromatic index.
- (c) Domination number.
- (d) Spectrum.

In Section 5.2 we study the chromatic number of s.c. chordal graphs. We prove that the chromatic number of s.c. chordal graphs with $4n$ vertices is $2n$ and that of s.c. chordal graphs with $4n+1$ vertices is $2n+1$. We also obtain bounds for the chromatic number of s.c.

perfect graphs and prove that the upper bounds are attained if and only if the graph is s.c. chordal. In Section 5.3 of Chapter 5 we study the chromatic index of s.c. chordal graphs. We give a sufficient condition for a s.c. chordal graph with $4n$ vertices to be a Class 1 graph and establish the Class 1 property for some classes of s.c. chordal graphs with $4n$ vertices. We also obtain bounds for the chromatic index of s.c. chordal graphs. In Section 5.4 we study the domination number of s.c. chordal graphs. In this we prove that the domination number and the independent domination number can be atmost $2n$ for s.c. chordal graphs with $4n$ vertices and atmost $2n+1$ for s.c. chordal graphs with $4n+1$ vertices. In Section 5.5 we discuss the spectrum of s.c. chordal graphs. In this we prove that the least positive integer for which there exist cospectral s.c. chordal graphs (s.c. chordal graphs with same spectrum) is 12. We also obtain bounds for the maximum eigenvalue of a s.c. chordal graph.

In Chapter 6 entitled ‘Conclusion’ we offer some research problems on s.c. chordal graphs.

1.3 Basic definitions and notations

In this thesis \square denotes the end of a proof. Let r be a real number. The greatest integer less than or equal to r and the smallest integer greater than or equal to r are denoted by $\lfloor r \rfloor$ and $\lceil r \rceil$ respectively. Let m, n and b be non-negative integers such that $n < b$. Then $m \equiv n \pmod{b}$ if $m = bj + n$ for some non-negative integer j . The sum of a sequence of n integers a_1, a_2, \dots, a_n is denoted by $\sum_{i=1}^n a_i$. The empty set is denoted by \emptyset . Let X be a set. The number of elements in X , an element x belongs to X and an element x does not belong to X are denoted by $|X|$, $x \in X$ and $x \notin X$ respectively. Let $x \in X$. The set $X - \{x\}$ is the set obtained from X by removing the element x . The union and the intersection of two sets X and Y are denoted by $X \cup Y$ and $X \cap Y$ respectively. A set Y that is contained in a set X is denoted by $Y \subseteq X$. Let $Y \subseteq X$. We denote $x_i \in Y$ implies $x_{i+1} \in Y$ for all $1 \leq i \leq n-1$ as $x_1 \Rightarrow x_2 \Rightarrow \dots \Rightarrow x_n \in Y$.

For the basic definitions and the notations related to the graphs we refer to [34], [101], [112] and [151].

Definition 1 : A graph G consists of a finite non-empty set $V(G)$ called the set of vertices

together with a prescribed set $E(G)$ of unordered pairs of distinct vertices of G . The number of vertices and the number of edges of a graph G are denoted by p and q respectively. Each pair $[u, v] \in E(G)$ is said to be incident with the vertices u and v and the vertices u and v are said to be adjacent. Two vertices u and v of G are non-adjacent if they are not adjacent. If two edges are incident with a common vertex then they are adjacent edges. Let v be a vertex of G and W be a subset of $V(G)$ not containing the vertex v . The set of all edges incident with v and a vertex in W is denoted by $[v, W]$. Let U and W be disjoint subsets of $V(G)$. The set of all edges incident with a vertex in U and a vertex in W is denoted by $[U, W]$.

Definition 2 : Let G be a graph. The degree of a vertex v in G is the number of edges incident with it and it is denoted by $\deg_G(v)$. The maximum degree of G is denoted by $\Delta(G)$. If all the vertices of a graph have the same degree then it is called a regular graph.

Definition 3 : The degree sequence of a graph G is the sequence of the degrees of the p vertices of G arranged in non-increasing order which is denoted by $d_1 \geq d_2 \geq \dots \geq d_p$.

Definition 4 : Let G be a graph. For a vertex v of G the set $Nhd_G(v)$ is the set of all vertices which are adjacent to v in G .

Definition 5 : A subgraph H of a graph G is a graph having all of its vertices and edges in G .

Definition 6 : Let G be a graph. For any set V' such that $V' \subseteq V(G)$, the vertex induced subgraph $\langle V' \rangle$ is the maximal subgraph of G with V' as the vertex set.

Definition 7 : Let G be a graph. For any set E' such that $E' \subseteq E(G)$, the edge induced subgraph $\langle E' \rangle$ is the subgraph of G with V' as its vertex set and E' as its edge set where V' consists of all the vertices incident with the edges in E' .

Definition 8 : A walk of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ beginning and ending with the vertices in which each edge is incident with the two vertices immediately preceding and following it. This walk joins v_0 and v_n . A walk is closed if $v_0 = v_n$. A walk is a path if all the vertices (and thus all the edges)

are distinct. A path consisting of n vertices is denoted by P_n . A closed path is called a cycle. A cycle consisting of n vertices is denoted by C_n . A vertex induced subgraph of a graph G which is a cycle is called an induced cycle of G .

Definition 9 : A graph is C_4 -free if it does not have any induced C_4 .

Definition 10 : A graph is connected if every pair of vertices is joined by a path. A maximal connected subgraph of a graph G is called a component of G .

Definition 11 : Let G be a graph. For a vertex v of G , the graph $G - v$ is obtained by deleting the vertex v and all the edges incident with it.

Definition 12 : Let G be a graph with $4n$ vertices. Let $U = \{v \in V(G) : \deg_G(v) \geq 2n\}$. Let u_0 be a vertex not belonging to $V(G)$. The graph $G_U \diamond u_0$ has $V(G) \cup \{u_0\}$ as its vertex set and $E(G) \cup E'$ as its edge set where E' is the set of edges obtained by joining u_0 to all the vertices of U .

Consider the graph G with 8 vertices shown in Figure 1.1(a). The graph $G_U \diamond u_0$ where $U = \{v \in V(G) : \deg_G(v) \geq 4\}$ and u_0 is a vertex not belonging to $V(G)$ is shown in Figure 1.1(b).

Definition 13 : A bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G is incident with a vertex of V_1 and a vertex of V_2 .

Definition 14 : A graph is said to be complete if every pair of its vertices are adjacent. A complete graph with p vertices is denoted by K_p . The graph $2K_2$ has two components each of which is K_2 . A clique of a graph is a maximal complete subgraph of the graph. A clique C of a graph G is maximum if there is no other clique of G with more number of vertices than the number of vertices in C . For a graph G the number of vertices in a maximum clique of G is called the clique number of G and it is denoted by $\omega(G)$.

Definition 15 : Let G be a graph. A subset S of G is a stable set (also called an independent set in the literature) if no two vertices of S are adjacent in G . A stable set of G which has

maximum number of vertices in it is called a maximum stable set of G . The number of vertices in a maximum stable set of G is called the stability number of G and it is denoted by $\alpha(G)$.

Definition 16 : A clique cover of order k of a graph G is a partition of its vertex set $V(G)$ into k subsets V_1, V_2, \dots, V_k such that each $\langle V_i \rangle$ is a clique of G . A clique cover of a graph G with smallest order is called a minimum clique cover of G . The order of a minimum clique cover of a graph G is called the clique cover number of G and it is denoted by $\theta(G)$.

Definition 17 : A proper vertex coloring of a graph G is assigning colors to the vertices of G such that no two adjacent vertices are assigned the same color. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to properly color the vertices of G .

Definition 18 : A proper edge coloring of a graph G is assigning colors to the edges of G such that no two adjacent edges are assigned the same color. The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors needed to properly color the edges of G . A graph G is said to be a Class 1 graph if $\chi'(G) = \Delta(G)$ and a Class 2 graph if $\chi'(G) = \Delta(G) + 1$.

Definition 19 : A graph G is χ -perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G . A graph G is α -perfect if $\alpha(H) = \theta(H)$ for every induced subgraph H of G . A graph G is perfect if it is χ -perfect (or equivalently it is α -perfect).

Definition 20 : A graph is chordal if it has no induced cycle C_n for all $n \geq 4$.

Definition 21 : A graph is split if its vertex set can be partitioned into V' and S such that $\langle V' \rangle$ is a clique of G and S is a stable set of G .

Definition 22 : Let G be a graph with p vertices whose vertices are labelled $1, 2, \dots, p$. For G its adjacency matrix is defined as the $p \times p$ matrix (a_{ij}) where a_{ij} is defined as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } [i, j] \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Definition 23 : *The eigen values of a graph G are the eigen values of the adjacency matrix of G obtained by labelling the vertices of G as $1, 2, \dots, p$. The eigenvalues of a graph constitute its spectrum. Two graphs are cospectral if they have the same spectrum. Maximum of the eigen values of a graph G is called the maximum eigen value of G and it is denoted by $\lambda_{\max}(G)$.*

Definition 24 : *The eigen vectors of a graph G are the eigen vectors of the adjacency matrix of G obtained by labelling the vertices of G as $1, 2, \dots, p$.*

Definition 25 : *Let G be a graph. A dominating set of G is a subset D of $V(G)$ such that each vertex of $V(G) - D$ is adjacent to atleast one vertex of D . A dominating set D of G is called a minimum dominating set if there is no other dominating set of G with lesser number of elements than D . The domination number $\delta_0(G)$ of G is the number of vertices in a minimum dominating set of G . A dominating set of G which is also a stable set is called a kernel of G . A kernel D of G is called a minimum kernel of G if there is no other kernel of G with lesser number of elements than D . The independent domination number $\delta_i(G)$ of G is the number of vertices in a minimum kernel of G .*

Definition 26 : *Two graphs G' and G'' are isomorphic if there exists a one-to-one correspondence between the vertex sets of G' and G'' which preserves adjacencies of the vertices. Such a one-to-one correspondence is called a vertex isomorphism of G' onto G'' and is denoted by ψ . If G' and G'' are isomorphic then we denote it by $G' \cong G''$. Two graphs are non-isomorphic if they are not isomorphic.*

Definition 27 : *Let G' and G'' be two isomorphic graphs and let ψ be an isomorphism of G' onto G'' . For every subset U of $V(G')$ the set $\psi(U) \subseteq V(G'')$ is defined as $\psi(U) = \{\psi(u) \in V(G'') : u \in U\}$.*

Definition 28 : *The problem of verifying whether two given graphs are isomorphic or not is called the isomorphism problem. The isomorphism problem of a class of graphs \mathcal{F} is called the isomorphism of \mathcal{F} .*

Definition 29 : *A graph parameter is a number or a set of numbers associated with graphs*

such that for any two isomorphic graphs the graph parameter is the same.

Definition 30 : The complement \bar{G} of a graph G has $V(G)$ as its vertex set and two vertices are adjacent in \bar{G} if and only if they are non-adjacent in G .

Definition 31 : A graph G is self-complementary (s.c.) if it is isomorphic to its complement \bar{G} . The number of non-isomorphic s.c. graphs with p vertices is denoted by s_p .

Definition 32 : A complementing permutation (c.p.) σ of a s.c. graph G is a vertex isomorphism of G onto \bar{G} . A c.p. σ of G can be written as product of disjoint permutation cycles denoted by $\sigma_1\sigma_2\cdots\sigma_s$ since G and \bar{G} have the same vertex set. Let $J \subseteq \{1, 2, \dots, s\}$. The product of the permutation cycles σ_i 's such that $i \in J$ is denoted by $\Pi_{i \in J} \sigma_i$. The product $\Pi_{i=2}^s \sigma_i$ is denoted by σ/σ_1 .

Definition 33 : Let G be a s.c. graph and let $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ where $\sigma_i = (v_{i1}v_{i2}\cdots v_{ip_i})$ be a c.p. of G . If σ has a single permutation cycle (that is when $s=1$) the vertices $v_{11}, v_{12}, \dots, v_{1p}$ are identified with the labels $1, 2, \dots, p$ respectively.

Definition 34 : Let G be a s.c. graph and $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ where $\sigma_i = (v_{i1}v_{i2}\cdots v_{ip_i})$ be a c.p. of G . A vertex v_{ij} of σ_i is even or odd labelled if the subscript j of v_{ij} is even or odd respectively. The set consisting of all even labelled vertices of σ_i and the set consisting of all odd labelled vertices of σ_i are denoted by $Even(\sigma_i)$ and $Odd(\sigma_i)$ respectively. The sets $Even(\sigma)$ and $Odd(\sigma)$ are defined to be $\cup_{i=1}^s Even(\sigma_i)$ and $\cup_{i=1}^s Odd(\sigma_i)$ respectively. The sets $Even(\sigma)$ and $Odd(\sigma)$ are called the set of even labelled vertices of σ and the set of odd labelled vertices of σ respectively.

Definition 35 : Let G be a s.c. graph and let $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ be a c.p. of G . The length of a permutation cycle σ_i is the number of vertices present in it.

Definition 36 : Let G be a s.c. graph and let σ be a c.p. of G . The vertex $\sigma(v)$ is the image of a vertex v under σ . The notation σ^k denotes σ multiplied k times. For a set $U \subseteq V(G)$, $\sigma^k(U)$ is defined as the set of all vertices $\sigma^k(u)$ such that $u \in U$.

Let G be a s.c. graph with a c.p. σ . Ringel [178] and Sachs [190] proved that the

length of every permutation cycle of σ is divisible by 4 when $p=4n$ and the length of every permutation cycle of σ is divisible by 4 except for a single permutation cycle of length 1 when $p=4n+1$.

Definition 37 : Let G be a s.c. graph with $p=4n$. A c.p. $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ of G where $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ such that $[v_{i2}, v_{i4}] \in E(G)$ for all $1 \leq i \leq s$ is called a star c.p. of G .

Consider the s.c. graph G shown in Figure 1.2. The permutation $(2\ 3\ 4\ 1)(5\ 6\ 7\ 8)$ on the vertices of G is a c.p. whereas it is not a star c.p. since $[3, 1] \notin E(G)$. The permutation $(1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8)$ is a star c.p. of G .

Definition 38 : Let G be a s.c. graph with $p=4n+1$ and a c.p. $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$. We assume σ_1 to be the permutation cycle of length 1 and denote the vertex present in σ_1 by v_0 . The vertex v_0 is called the fixed vertex of σ .

Definition 39 : Let G be a s.c. graph with $p=4n+1$. A c.p. $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ of G where $\sigma_1^* = (v_0)$ and $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ for all $2 \leq i \leq s$ is called a star c.p. of G if $[v_{i2}, v_{i4}] \in E(G)$ for all $2 \leq i \leq s$.

Consider the s.c. graph G shown in Figure 1.3. The permutation $(0)(2\ 3\ 4\ 1)(5\ 6\ 7\ 8)$ on the vertices of G is a c.p. whereas it is not a star c.p. since $[3, 1] \notin E(G)$. The permutation $(0)(1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8)$ is a star c.p. of G .

Remark: In Section 2.3 we prove that every s.c. graph has a star c.p. (Theorem 2.6).

Definition 40 : A s.c. graph which is chordal is called a s.c. chordal graph.

Definition 41 : A s.c. graph which is perfect is called a s.c. perfect graph.

Definition 42 : A list of all non-isomorphic s.c. graphs with p vertices is called the catalogue of s.c. graphs with p vertices. A list of all non-isomorphic s.c. chordal graphs with p vertices is called the catalogue of s.c. chordal graphs with p vertices.

Definition 43 : An algorithm is said to run in $O(f(x))$ time if the time taken by the algorithm is at most $cf(x)$ where c is a positive integer and x is the size of the input. In this

thesis for a s.c. chordal graph with $4n$ or $4n+1$ vertices we always take n to be the size of the input.

Definition 44 : An algorithm is said to be polynomially computable if the running time of the algorithm is $O(p(x))$ where $p(x)$ is a polynomial function. An algorithm is said to be computable in linear time if the running time of the algorithm is $O(p(x))$ where $p(x)$ is a polynomial of degree 1.

Definition 45 : The isomorphism of a class of graphs \mathcal{F}_1 is said to be polynomially reducible to the isomorphism of a class of graphs \mathcal{F}_2 if the following hold.

- (i) There exists a polynomially computable algorithm which constructs a graph G' of \mathcal{F}_2 from a graph G of \mathcal{F}_1 .
- (ii) Two graphs G_1 and G_2 of \mathcal{F}_1 are isomorphic if and only if the graphs G'_1 and G'_2 of \mathcal{F}_2 (constructed by the polynomially computable algorithm) are isomorphic.

Definition 46 : The isomorphism of two classes of graphs \mathcal{F}_1 and \mathcal{F}_2 are polynomially equivalent if the isomorphism of \mathcal{F}_1 is polynomially reducible to the isomorphism of \mathcal{F}_2 and vice versa.

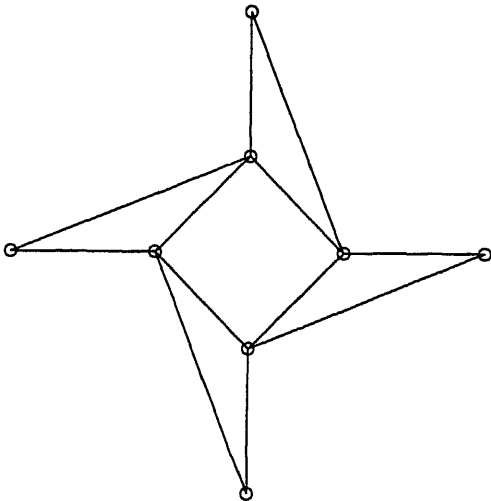


Figure 1.1a

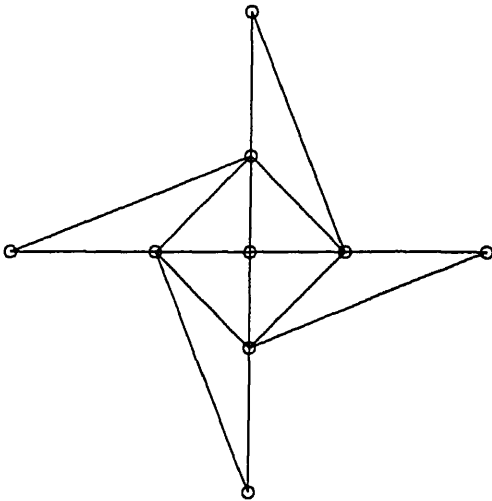


Figure 1.1b

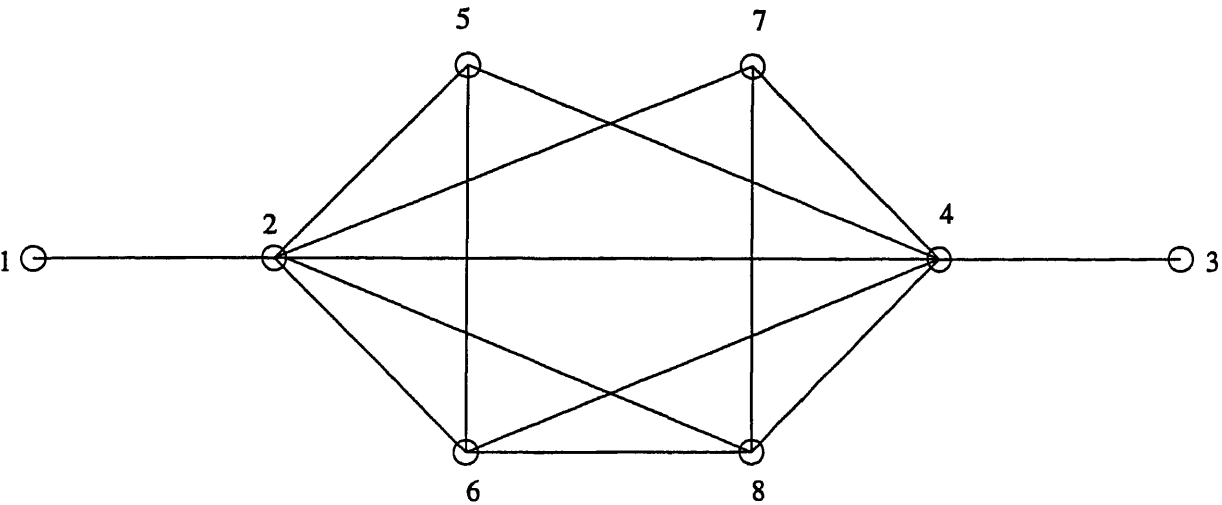


Figure 1.2

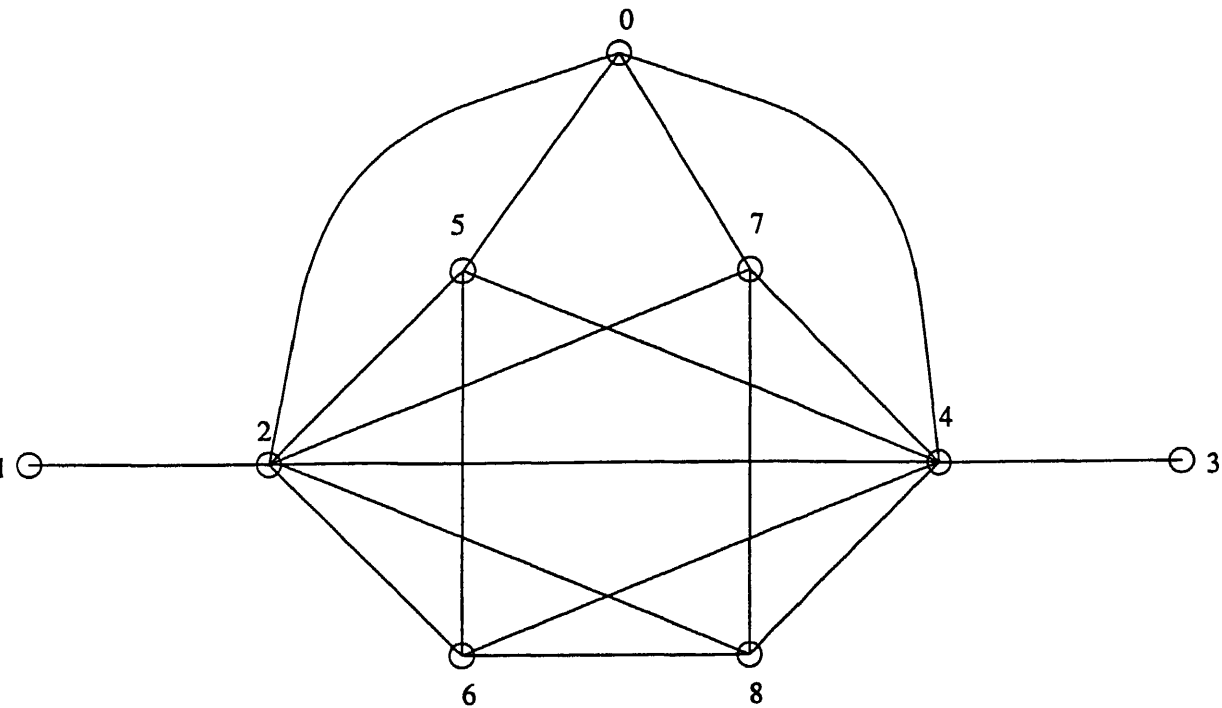


Figure 1.3

Chapter 2

On the characterisations for self-complementary graphs to be chordal

2.1 Introduction

In the literature various characterisations were obtained for chordal graphs and some classes of chordal graphs like split graphs, interval graphs etc. For these we refer to [10], [24], [39], [75], [85], [89], [95], [101], [109] and [229]. In this chapter we obtain the following characterisations.

- (i) Let G be a s.c. graph. Then G is chordal if and only if $\omega(G) = 2n$ when $p = 4n$ and $\omega(G) = 2n + 1$ when $p = 4n + 1$ where p denotes the number of vertices of G and $\omega(G)$ denotes the clique number of G .
- (ii) Let G be a s.c. graph. Then G is chordal if and only if $\alpha(G) = 2n$ when $p = 4n$ and $\alpha(G) = 2n + 1$ when $p = 4n + 1$ where p denotes the number of vertices of G and $\alpha(G)$ denotes the stability number of G .
- (iii) Let G be a s.c. graph. Then G is chordal if and only if G has no induced subgraph isomorphic to C_4 when $p = 4n$ and G has no induced subgraph isomorphic to C_4 or

C_5 when $p = 4n + 1$ where p denotes the number of vertices of G and C_n denotes the cycle with n vertices.

- (iv) Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then G is chordal if and only if $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ when $p = 4n$ and $\sum_{i=1}^{2n} d_i = 6n^2$ when $p = 4n + 1$ where p denotes the number of vertices of G .

The characterisations (i), (ii), (iii) and (iv) are obtained in Section 2.3, Section 2.4, Section 2.5 and Section 2.6 respectively by using the results stated in Section 2.2.

2.2 Basic Results

For obtaining the various characterisations for a self-complementary graph to be a chordal graph we use the following results available in the literature.

Ringel [178] and Sachs [190] independently obtained the following result on the structure of a c.p. of a s.c. graph.

Theorem 2.1 (Ringel [178], Sachs [190]) : *Let G be a s.c. graph with a c.p. σ . Then*

- (i) *the order of every permutation cycle of σ is divisible by 4 if $p=4n$*
- (ii) *the order of every permutation cycle of σ is divisible by 4, except for a single permutation cycle of order 1 if $p=4n+1$.*

Foldes and Hammer [85] had obtained the following characterisations for split graphs.

Theorem 2.2 (Foldes and Hammer [85]) : *Let G be a graph. Then G is a split graph if and only if G has no induced subgraph isomorphic to $2K_2$, C_4 or C_5 .*

Theorem 2.3 (Foldes and Hammer [85]) : *Let G be a graph. Then G is a split graph if and only if both G and \bar{G} are chordal graphs.*

The following results are due to Hammer and Simone [109].

Theorem 2.4 (Hammer and Simone [109]) : Let G be a graph with the degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Let $\mathcal{M} = \max\{i : d_i \geq i - 1, 1 \leq i \leq p\}$. Then G is a split graph implies $\omega(G) = \mathcal{M}$.

Theorem 2.5 (Hammer and Simone [109]) : Let G be a graph with the degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Let $\mathcal{M} = \max\{i : d_i \geq i - 1, 1 \leq i \leq p\}$. Then G is a split graph if and only if $\sum_{i=1}^{\mathcal{M}} d_i = \mathcal{M}(\mathcal{M} - 1) + \sum_{i=\mathcal{M}+1}^p d_i$.

2.3 A characterisation for self-complementary graphs to be chordal in terms of the clique number

Lemma 2.1 : Let G be a s.c. graph with a c.p. σ . Let u and v be two vertices of G . Then $[u, v] \in E(G)$ if and only if $[\sigma(u), \sigma(v)] \notin E(G)$.

Proof: We note that $[u, v] \in E(G)$ if and only if $[\sigma(u), \sigma(v)] \notin E(\bar{G})$ for σ is an isomorphism of G onto \bar{G} . Also $[\sigma(u), \sigma(v)] \in E(\bar{G})$ if and only if $[\sigma(u), \sigma(v)] \notin E(G)$. Hence $[u, v] \in E(G)$ if and only if $[\sigma(u), \sigma(v)] \notin E(G)$. \square

Lemma 2.2 : Let G be a s.c. graph with a c.p. σ . Let u and v be two vertices of G . Then

- (i) $[u, v] \in E(G)$ if and only if $[\sigma^{2i}(u), \sigma^{2i}(v)] \in E(G)$
- (ii) $[u, v] \in E(G)$ if and only if $[\sigma^{2i+1}(u), \sigma^{2i+1}(v)] \notin E(G)$

where i is a positive integer.

Proof: (i) Let u and v be two vertices of G . We prove that $[u, v] \in E(G)$ if and only if $[\sigma^{2i}(u), \sigma^{2i}(v)] \in E(G)$ for all positive integers i , by induction on i . By Lemma 2.1 $[u, v] \in E(G)$ if and only if $[\sigma(u), \sigma(v)] \notin E(G)$. Also by Lemma 2.1 $[\sigma(u), \sigma(v)] \notin E(G)$ if and only if $[\sigma^2(u), \sigma^2(v)] \in E(G)$. Hence $[u, v] \in E(G)$ if and only if $[\sigma^2(u), \sigma^2(v)] \in E(G)$. Therefore for $i = 1$ the result is true. Assume the result for $i = k$. That is $[u, v] \in E(G)$ if and only if $[\sigma^k(u), \sigma^k(v)] \in E(G)$. By Lemma 2.1 $[\sigma^{2k}(u), \sigma^{2k}(v)] \in E(G)$ if and only if $[\sigma^{2k+1}(u), \sigma^{2k+1}(v)] \notin E(G)$. Also by Lemma 2.1 $[\sigma^{2k+1}(u), \sigma^{2k+1}(v)] \notin E(G)$ if and

only if $[\sigma^{2k+2}(u), \sigma^{2k+2}(v)] \in E(G)$. Hence by the assumption $[u, v] \in E(G)$ if and only if $[\sigma^{2k+2}(u), \sigma^{2k+2}(v)] \in E(G)$. Therefore the result is true for $i = k + 1$.

(ii) By Lemma 2.1 $[u, v] \in E(G)$ if and only if $[\sigma(u), \sigma(v)] \notin E(G)$. By Lemma 2.2(i) $[\sigma(u), \sigma(v)] \notin E(G)$ if and only if $[\sigma^{2i+1}(u), \sigma^{2i+1}(v)] \notin E(G)$ where i is a positive integer. Hence $[u, v] \in E(G)$ if and only if $[\sigma^{2i+1}(u), \sigma^{2i+1}(v)] \notin E(G)$ where i is a positive integer. \square

Existence of a star c.p. for every s.c. graph follows from the following result.

Theorem 2.6 : *Let G be a s.c. graph. There exists a star c.p. σ^* of G .*

Proof: Let $p=4n$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$ be a c.p. of G where $\sigma_i = (v_{i1} v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$. By Lemma 2.1 either $[v_{i1}, v_{i3}] \in E(G)$ or $[\sigma(v_{i1}), \sigma(v_{i3})] = [v_{i2}, v_{i4}] \in E(G)$ for all $1 \leq i \leq s$. If $[v_{i2}, v_{i4}] \in E(G)$ for all $1 \leq i \leq s$, relabel the vertices $v_{i1}, v_{i2}, \dots, v_{ip_i}$ as $v_{i1}^*, v_{i2}^*, \dots, v_{ip_i}^*$ respectively, for all $1 \leq i \leq s$. Otherwise relabel the vertices $v_{i1}, v_{i2}, \dots, v_{i(p_i-1)}, v_{ip_i}$ as $v_{i2}^*, v_{i3}^*, \dots, v_{ip_i}^*, v_{i1}^*$ respectively, for all $1 \leq i \leq s$. Define $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1}^* v_{i2}^* \cdots v_{ip_i}^*)$ for all $1 \leq i \leq s$. We note that $[v_{i2}^*, v_{i4}^*] \in E(G)$ for all $1 \leq i \leq s$ since $[v_{i2}^*, v_{i4}^*] = [v_{i2}, v_{i4}]$ if $[v_{i2}, v_{i4}] \in E(G)$ and $[v_{i2}^*, v_{i4}^*] = [v_{i1}, v_{i3}]$ if $[v_{i1}, v_{i3}] \in E(G)$ for all $1 \leq i \leq s$. Also σ^* is a c.p. of G since σ is a c.p. of G .

Let $p=4n+1$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$ be a c.p. of G where $\sigma_1 = (v_0)$ and $\sigma_i = (v_{i1} v_{i2} \cdots v_{ip_i})$ for all $2 \leq i \leq s$. By Lemma 2.1 either $[v_{i1}, v_{i3}] \in E(G)$ or $[\sigma(v_{i1}), \sigma(v_{i3})] = [v_{i2}, v_{i4}] \in E(G)$ for all $2 \leq i \leq s$. If $[v_{i2}, v_{i4}] \in E(G)$ for all $2 \leq i \leq s$, relabel the vertices $v_{i1}, v_{i2}, \dots, v_{ip_i}$ as $v_{i1}^*, v_{i2}^*, \dots, v_{ip_i}^*$ respectively, for all $2 \leq i \leq s$. Otherwise relabel the vertices $v_{i1}, v_{i2}, \dots, v_{i(p_i-1)}, v_{ip_i}$ as $v_{i2}^*, v_{i3}^*, \dots, v_{ip_i}^*, v_{i1}^*$ respectively, for all $2 \leq i \leq s$. Define $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1}^* v_{i2}^* \cdots v_{ip_i}^*)$ for all $2 \leq i \leq s$. We note that $[v_{i2}^*, v_{i4}^*] \in E(G)$ for all $2 \leq i \leq s$ since $[v_{i2}^*, v_{i4}^*] = [v_{i2}, v_{i4}]$ if $[v_{i2}, v_{i4}] \in E(G)$ and $[v_{i2}^*, v_{i4}^*] = [v_{i1}, v_{i3}]$ if $[v_{i1}, v_{i3}] \in E(G)$ for all $2 \leq i \leq s$. Also σ^* is a c.p. of G since σ is a c.p. of G . \square

The following result determines the degree sequence of the complement of a graph G from the degree sequence of G .

Theorem 2.7 : *Let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_p$. Then the degree sequence of \bar{G} is $p-1-d_p \geq p-1-d_{p-1} \geq \cdots \geq p-1-d_1$.*

Proof: Let v be a vertex of G with degree d . Then $p-1-d$ is the degree of v in \tilde{G} . \square

The following result establishes a relation between the i th term and the $(p-i+1)$ th term of the degree sequence of a s.c. graph.

Theorem 2.8 : *Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then $d_i + d_{p-i+1} = p-1$ where $1 \leq i \leq p$.*

Proof: By Theorem 2.7 $p-1-d_p \geq p-1-d_{p-1} \geq \dots \geq p-1-d_1$ is the degree sequence of G since G is a s.c. graph. Hence $d_i = p-1-d_{p-i+1}$ for all $1 \leq i \leq p$. \square

For a s.c. graph G with the degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$ the following result determines the value of \mathcal{M} where $\mathcal{M} = \max\{i : d_i \geq i-1, 1 \leq i \leq p\}$.

Theorem 2.9 : *Let G be a s.c. graph with the degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Let $\mathcal{M} = \max\{i : d_i \geq i-1, 1 \leq i \leq p\}$. Then*

(i) $\mathcal{M} = 2n$ if $p=4n$

(ii) $\mathcal{M} = 2n+1$ if $p=4n+1$.

Proof: (i) Let $p=4n$. We prove that $d_i \geq 2n$ for all $1 \leq i \leq 2n$ and $d_i \leq 2n-1$ for all $2n+1 \leq i \leq 4n$. Let $d_k \leq 2n-1$ for some $1 \leq k \leq 2n$. By Theorem 2.8 $d_{p-k+1} \geq 2n$ since $d_k \leq 2n-1$. Also $d_k \geq d_{p-k+1}$ since $1 \leq k \leq 2n$. Hence $2n-1 \geq 2n$, a contradiction. Therefore $d_i \geq 2n$ for all $1 \leq i \leq 2n$. Let $d_k \geq 2n-1$ for some $2n+1 \leq k \leq 4n$. By Theorem 2.8 $d_{p-k+1} \leq 2n-1$ since $d_k \geq 2n-1$. Also $d_{p-k+1} \geq d_k$ since $2n+1 \leq k \leq 4n$. Hence $2n-1 \geq 2n$, a contradiction. Therefore $d_i \leq 2n-1$ for all $2n+1 \leq i \leq 4n$. We note that $\mathcal{M} = \max\{i : d_i \geq i-1, 1 \leq i \leq p\} = 2n$ since $d_i \geq 2n$ for all $1 \leq i \leq 2n$ and $d_i \leq 2n-1$ for all $2n+1 \leq i \leq 4n$.

(ii) Let $p=4n+1$. We prove that $d_i \geq 2n$ for all $1 \leq i \leq 2n+1$ and $d_i \leq 2n$ for all $2n+1 \leq i \leq 4n+1$. Let $d_k \leq 2n-1$ for some $1 \leq k \leq 2n+1$. By Theorem 2.8 $d_{p-k+1} \geq 2n+1$ since $d_k \leq 2n-1$. Also $d_{p-k+1} \leq d_k$ since $1 \leq k \leq 2n+1$. Hence $2n-1 \geq 2n+1$, a contradiction. Therefore $d_i \geq 2n$ for all $1 \leq i \leq 2n+1$. Let $d_k \geq 2n+1$ for some $2n+1 \leq k \leq 4n+1$. By Theorem 2.8 $d_{p-k+1} \leq 2n-1$ since $d_k \geq 2n+1$. Also $d_k \leq d_{p-k+1}$ since $2n+1 \leq k \leq 4n+1$. Hence $2n-1 \geq 2n+1$, a contradiction. Therefore $d_i \leq 2n$ for

all $2n + 1 \leq i \leq 4n + 1$. We note that $\mathcal{M} = \max\{i : d_i \geq i - 1, 1 \leq i \leq p\} = 2n + 1$. \square

The following result gives a necessary condition for a s.c. graph to be a split graph in terms of its clique number.

Theorem 2.10 : *Let G be a s.c. graph. Then G is a split graph implies*

(i) $\omega(G) = 2n$ if $p = 4n$

(ii) $\omega(G) = 2n + 1$ if $p = 4n + 1$.

Proof: Follows from Theorem 2.4 and Theorem 2.9. \square

Lemma 2.3 : *Let G be a s.c. graph. Let H be an induced subgraph of G . Then there exists an induced subgraph of G isomorphic to \bar{H} .*

Proof: Let σ be a c.p. of G . Let V_H be the vertex set of H . We note that $\langle \sigma(V_H) \rangle$ is an induced subgraph of G isomorphic to \bar{H} since σ is a c.p. \square

An upper bound for the clique number of a s.c. graph is obtained in the following Theorem.

Theorem 2.11 : *Let G be a s.c. graph. Then*

(i) $\omega(G) \leq 2n$ if $p = 4n$

(ii) $\omega(G) \leq 2n + 1$ if $p = 4n + 1$.

Proof: (i) Let $p = 4n$. Let $\omega(G) \geq 2n + 1$. Let C be a maximum clique of G . Let K be a complete subgraph of C with $2n + 1$ vertices. By Lemma 2.3 there exists a subgraph S of G isomorphic to \bar{K} . Let V_K be the vertex set of K . We note that V_K and S can have at most one common vertex since K is a complete subgraph of G and S is a stable set of G . Hence $|V_K \cup S| \geq 4n + 1$, thus a contradiction since $|V_K \cup S| \leq 4n$. Therefore $\omega(G) \leq 2n$.

(ii) Let $p = 4n + 1$. Let $\omega(G) \geq 2n + 2$. Let C be a maximum clique of G . Let K be a complete subgraph of C with $2n + 2$ vertices. By Lemma 2.3 there exists a subgraph S of G isomorphic to \bar{K} . Let V_K be the vertex set of K . We note that V_K and S can have at most one common vertex since K is a complete subgraph of G and S is a stable set of G . Hence $|V_K \cup S| \geq 4n + 3$, thus a contradiction since $|V_K \cup S| \leq 4n + 1$. Therefore $\omega(G) \leq 2n + 1$. \square

Theorem 2.12 : Let G be a s.c. graph with a c.p. $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ where $\sigma_i = (v_{i1}v_{i2}\cdots v_{ip_i})$ for all $1 \leq i \leq s$. Let $X_i = \{v_{ij} \in \sigma_i\}$ where $1 \leq i \leq s$ and let $Y_0 = \{1, 2, \dots, s\}$. Then $\langle \bigcup_{i \in J} X_i \rangle$ is a s.c. graph with a c.p. $\sigma_J = \Pi_{i \in J} \sigma_i$ where $J \subseteq Y_0$.

Proof: Choose $J \subseteq Y_0 = \{1, 2, \dots, s\}$. We note that $\sigma_J = \Pi_{i \in J} \sigma_i$ is a c.p. of $\langle \bigcup_{i \in J} X_i \rangle$ since σ is a c.p. of G . Hence $\langle \bigcup_{i \in J} X_i \rangle$ is a s.c. graph. \square

Corollary 2.1 : Let G be a s.c. graph with $p=4n+1$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ be a c.p. of G with v_0 as its fixed vertex. Then $G - v_0$ is a s.c. graph with a c.p. σ/σ_1 .

Proof: Follows from Theorem 2.12. \square

Theorem 2.13 : Let G be a s.c. graph with a star c.p. $\sigma^* = \sigma_1^*\sigma_2^*\cdots\sigma_s^*$ where $\sigma_i^* = (v_{i1}v_{i2}\cdots v_{ip_i})$ for all $1 \leq i \leq s$. Let $X_i = \{v_{ij} \in \sigma_i^*\}$ where $1 \leq i \leq s$ and let $Y_0 = \{1, 2, \dots, s\}$. Then $\langle \bigcup_{i \in J} X_i \rangle$ is a s.c. graph with a star c.p. $\sigma_J^* = \Pi_{i \in J} \sigma_i^*$ where $J \subseteq Y_0$.

Proof: Follows from the definition of star c.p. of a s.c. graph and Theorem 2.12. \square

Corollary 2.2 : Let G be a s.c. graph with $p=4n+1$. Let $\sigma^* = \sigma_1^*\sigma_2^*\cdots\sigma_s^*$ be a star c.p. of G with v_0 as its fixed vertex. Then $G - v_0$ is a s.c. graph with a star c.p. σ^*/σ_1^* .

Proof: Follows from Theorem 2.13. \square

Lemma 2.4 : Let G be a s.c. graph with a c.p. $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ where $\sigma_i = (v_{i1}v_{i2}\cdots v_{ip_i})$ for all $1 \leq i \leq s$. Let V' be a subset of V such that $\langle V' \rangle$ is a complete subgraph of G . If $v_{im} \in V'$ and $\sigma(v_{im}) \in V'$ for some vertex v_{im} of G then either $v_{jn} \notin V'$ or $\sigma(v_{jn}) \notin V'$ for every vertex v_{jn} of G distinct from v_{im} where $1 \leq i \leq s$, $1 \leq j \leq s$, $1 \leq m \leq p_i$ and $1 \leq n \leq p_j$.

Proof: Let v_{im} , $\sigma(v_{im})$, v_{jn} and $\sigma(v_{jn})$ belong to V' where v_{im} and v_{jn} are two distinct vertices of G such that and $1 \leq i \leq s$, $1 \leq j \leq s$, $1 \leq m \leq p_i$ and $1 \leq n \leq p_j$. Then $[v_{im}, v_{jn}] \in E(G)$ and $[\sigma(v_{im}), \sigma(v_{jn})] \in E(G)$ since $\langle V' \rangle$ is a complete subgraph of G . By Lemma 2.1 $[\sigma(v_{im}), \sigma(v_{jn})] \notin E(G)$ since $[v_{im}, v_{jn}] \in E(G)$, a contradiction. \square

Lemma 2.5 : Let G be a s.c. graph with $p=4n$, $\omega(G) = 2n$ and a c.p. $\sigma = (1\ 2\cdots 4n)$

where $n \geq 2$. Let C be a maximum clique of G and V' be the vertex set of C . Then there exists no vertex i of G such that $i \in V'$ and $\sigma(i) \in V'$ where $1 \leq i \leq 4n$.

Proof: Let $k \in V'$ and $\sigma(k) \in V'$ for some vertex $1 \leq k \leq 4n$. We note that $[k, \sigma(k)] \in E(G)$ since $k \in V'$ and $\sigma(k) \in V'$. By Lemma2.1 $[\sigma(k), \sigma^2(k)] \notin E(G)$ and by Lemma2.2 $[\sigma^{4n-1}(k), \sigma^{4n}(k)] = [\sigma^{4n-1}(k), k] \notin E(G)$ since $[k, \sigma(k)] \in E(G)$. Hence $\sigma^2(k) \notin V'$ and $\sigma^{4n-1}(k) \notin V'$. By Lemma2.2 either $[\sigma(k), \sigma^3(k)] \notin E(G)$ or $[\sigma^{4n-2}(k), \sigma^{4n}(k)] = [\sigma^{4n-2}(k), k] \notin E(G)$. Hence either $\sigma^3(k) \notin V'$ or $\sigma^{4n-2}(k) \notin V'$ since $k \in V'$ and $\sigma(k) \in V'$. By Lemma 2.4 atmost one vertex of each set $\{\sigma^4(k), \sigma^5(k)\}, \{\sigma^6(k), \sigma^7(k)\}, \dots, \{\sigma^{4n-4}(k), \sigma^{4n-3}(k)\}$ belongs to V' since $k \in V'$ and $\sigma(k) \in V'$. Hence either $\sigma^3(k)$ or $\sigma^{4n-2}(k)$ but not both and exactly one vertex of each set $\{\sigma^4(k), \sigma^5(k)\}, \{\sigma^6(k), \sigma^7(k)\}, \dots, \{\sigma^{4n-4}(k), \sigma^{4n-3}(k)\}$ belongs to V' since $|V'| = 2n$. Let $\sigma^3(k) \in V'$. By Lemma2.2 $[\sigma^3(k), \sigma^4(k)] \notin E(G)$, $[\sigma^5(k), \sigma^6(k)] \notin E(G), \dots, [\sigma^{4n-5}(k), \sigma^{4n-4}(k)] \notin E(G)$ since $[k, \sigma(k)] \in E(G)$. Hence $\sigma^3(k) \Rightarrow \sigma^5(k) \Rightarrow \sigma^7(k) \Rightarrow \dots \Rightarrow \sigma^{4n-3}(k) \in V'$. Therefore $\sigma^3(k) \in V'$ and $\sigma^{4n-3}(k) \in V'$. By Lemma2.2 $[k, \sigma^3(k)] \notin E(G)$ or $[\sigma^{4n-3}(k), \sigma^{4n}(k)] = [\sigma^{4n-3}(k), k] \notin E(G)$. Hence $\sigma^3(k) \notin V'$ or $\sigma^{4n-3}(k) \notin V'$ since $k \in V'$, thus a contradiction to the fact that both $\sigma^3(k) \in V'$ and $\sigma^{4n-3}(k) \in V'$. Therefore $\sigma^3(k) \notin V'$. Then $\sigma^{4n-2}(k) \in V'$ since exactly one vertex of the vertices $\sigma^3(k)$ and $\sigma^{4n-2}(k)$ belong to V' . By Lemma2.2 $[\sigma^5(k), \sigma^6(k)] \notin E(G)$, $[\sigma^7(k), \sigma^8(k)] \notin E(G), \dots, [\sigma^{4n-3}(k), \sigma^{4n-2}(k)] \notin E(G)$ since $[k, \sigma(k)] \notin E(G)$. Hence $\sigma^{4n-2}(k) \Rightarrow \sigma^{4n-4}(k) \Rightarrow \dots \Rightarrow \sigma^4(k) \in V'$. Therefore $\sigma^4(k) \in V'$ and $\sigma^{4n-2}(k) \in V'$. By Lemma 2.2 $[\sigma(k), \sigma^4(k)] \notin E(G)$ or $[\sigma^{4n-2}(k), \sigma^{4n+1}(k)] = [\sigma^{4n-2}(k), \sigma(k)] \notin E(G)$. Hence either $\sigma^4(k) \notin V'$ or $\sigma^{4n-2}(k) \notin V'$ since $\sigma(k) \in V'$, thus a contradiction to the fact that both $\sigma^4(k)$ and $\sigma^{4n-2}(k)$ belong to V' . Therefore there exists no vertex i of G such that $i \in V'$ and $\sigma(i) \in V'$ for all $1 \leq i \leq 4n$. \square

Lemma 2.6 : Let G be a s.c. graph with $p=4n$, $\omega(G) = 2n$ and a c.p. $\sigma = (1\ 2 \dots 4n)$ where $n \geq 2$. Let C be a maximum clique of G and V' be the vertex set of C . Then either $V' = \text{Even}(\sigma)$ or $V' = \text{Odd}(\sigma)$.

Proof: By Lemma 2.1 either $[1, 3] \notin E(G)$ or $[\sigma(1), \sigma(3)] = [2, 4] \notin E(G)$.

Case i : We prove that $V' = \text{Even}(\sigma)$ when $[1, 3] \notin E(G)$. Let $[1, 3] \notin E(G)$. We note that

$|V'| = 2n$ since $\omega(G) = 2n$. Also $|Even(\sigma)| = 2n$ and $Even(\sigma) \cap Odd(\sigma) = \emptyset$. Hence to prove $V' = Even(\sigma)$ we prove that V' does not have any odd labelled vertex. Let an odd labelled vertex $2r+1 \in V'$ for some $0 \leq r \leq 2n-1$. By Lemma2.2 either $[2r+1, \sigma(2r+1)] \in E(G)$ or $[\sigma^{4n-1}(2r+1), \sigma^{4n}(2r+1)] = [\sigma^{4n-1}(2r+1), 2r+1] \in E(G)$. Let $[2r+1, \sigma(2r+1)] \in E(G)$. By Lemma2.5 exactly one vertex of each set $\{2r+1, \sigma(2r+1)\}, \{\sigma^2(2r+1), \sigma^3(2r+1)\}, \dots, \{\sigma^{4n-2}(2r+1), \sigma^{4n-1}(2r+1)\}$ belong to V' since $|V'| = 2n$ and $\sigma^{4n}(2r+1) = 2r+1$. By Lemma2.2 $[\sigma^3(2r+1), \sigma^4(2r+1)] \notin E(G), [\sigma^5(2r+1), \sigma^6(2r+1)] \notin E(G), \dots, [\sigma^{4n-1}(2r+1), \sigma^{4n}(2r+1)] = [\sigma^{4n-1}(2r+1), 2r+1] \notin E(G)$, since $[2r+1, \sigma(2r+1)] \in E(G)$. Hence $2r+1 \Rightarrow \sigma^{4n-2}(2r+1) \Rightarrow \sigma^{4n-4}(2r+1) \Rightarrow \dots \Rightarrow \sigma^2(2r+1) \in V'$. Therefore the $2n$ vertices of C are $2r+1, \sigma^2(2r+1), \dots, \sigma^{4n-2}(2r+1)$ since the vertices $2r+1, \sigma^2(2r+1), \dots, \sigma^{4n-2}(2r+1)$ are mutually distinct. Also $\sigma^{2i}(2r+1)$ is odd labelled where $1 \leq i \leq 2n-1$. Hence $V' = Odd(\sigma)$ since $|Odd(\sigma)| = 2n$, thus a contradiction to the fact that $[1, 3] \notin E(G)$. Therefore $[2r+1, \sigma(2r+1)] \notin E(G)$. Then $[\sigma^{4n-1}(2r+1), 2r+1] \in E(G)$ since either $[2r+1, \sigma(2r+1)] \in E(G)$ or $[\sigma^{4n-1}(2r+1), 2r+1] \in E(G)$. By Lemma 2.5 exactly one vertex of each set $\{\sigma(2r+1), \sigma^2(2r+1)\}, \{\sigma^3(2r+1), \sigma^4(2r+1)\}, \dots, \{\sigma^{4n-1}(2r+1), \sigma^{4n}(2r+1)\} = \{\sigma^{4n-1}(2r+1), 2r+1\}$ belong to V' since $|V'| = 2n$. By Lemma2.2 $[2r+1, \sigma(2r+1)] \notin E(G), [\sigma^2(2r+1), \sigma^3(2r+1)] \notin E(G), \dots, [\sigma^{4n-2}(2r+1), \sigma^{4n-1}(2r+1)] \notin E(G)$ since $[\sigma^{4n-1}(2r+1), 2r+1] \in E(G)$ and $\sigma^{4n}(2r+1) = 2r+1$. Hence $2r+1 \Rightarrow \sigma^2(2r+1) \Rightarrow \sigma^4(2r+1) \Rightarrow \dots \Rightarrow \sigma^{4n-2}(2r+1) \in V'$. Therefore the $2n$ vertices of C are $2r+1, \sigma^2(2r+1), \dots, \sigma^{4n-2}(2r+1)$ since the vertices $2r+1, \sigma^2(2r+1), \dots, \sigma^{4n-2}(2r+1)$ are mutually distinct. Also $\sigma^{2i}(2r+1)$ is odd labelled where $1 \leq i \leq 2n-1$. Hence $V' = Odd(\sigma)$ since $|Odd(\sigma)| = 2n$, thus a contradiction to the fact that $[1, 3] \notin E(G)$. Therefore V' does not have any odd labelled vertex. Hence $V' = Even(\sigma)$ when $[1, 3] \notin E(G)$.

Case ii : We prove that $V' = Odd(\sigma)$ when $[2, 4] \notin E(G)$. Let $[2, 4] \notin E(G)$. We note that $|V'| = 2n$ since $\omega(G) = 2n$. Also $|Odd(\sigma)| = 2n$ and $Odd(\sigma) \cap Even(\sigma) = \emptyset$. Hence to show $V' = Odd(\sigma)$ we prove that V' does not have any even labelled vertex. Let an even labelled vertex $2r \in V'$ for some $1 \leq r \leq 2n$. By Lemma2.2 either $[2r, \sigma(2r)] \in E(G)$ or $[\sigma^{4n-1}(2r), \sigma^{4n}(2r)] = [\sigma^{4n-1}(2r), 2r] \in E(G)$. Let $[2r, \sigma(2r)] \in E(G)$. By Lemma 2.5 exactly one vertex of each set $\{2r, \sigma(2r)\}, \{\sigma^2(2r), \sigma^3(2r)\}, \dots, \{\sigma^{4n-2}(2r), \sigma^{4n-1}(2r)\}$ belongs to V'

since $|V'| = 2n$ and $\sigma^{4n}(2r) = 2r$. By Lemma2.2 $[\sigma^3(2r), \sigma^4(2r)] \notin E(G), [\sigma^5(2r), \sigma^6(2r)] \notin E(G), \dots, [\sigma^{4n-1}(2r), \sigma^{4n}(2r)] = [\sigma^{4n-1}(2r), 2r] \notin E(G)$ since $[2r, \sigma(2r)] \in E(G)$. Hence $2r \Rightarrow \sigma^{4n-2}(2r) \Rightarrow \sigma^{4n-4}(2r) \Rightarrow \dots \Rightarrow \sigma^2(2r) \in V'$. Therefore the $2n$ vertices of C are $2r, \sigma^2(2r), \dots, \sigma^{4n-2}(2r)$ since the vertices $2r, \sigma^2(2r), \dots, \sigma^{4n-2}(2r)$ are mutually distinct. Also $\sigma^{2i}(2r)$ is even labelled where $1 \leq i \leq 2n - 1$. Hence $V' = \text{Even}(\sigma)$ since $|\text{Even}(\sigma)| = 2n$, thus a contradiction to the fact that $[2, 4] \notin E(G)$. Therefore $[2r, \sigma(2r)] \notin E(G)$. Then $[\sigma^{4n-1}(2r), 2r] \in E(G)$ since either $[2r, \sigma(2r)] \in E(G)$ or $[\sigma^{4n-1}(2r), 2r] \in E(G)$. By Lemma 2.5 exactly one vertex of each set $\{\sigma(2r), \sigma^2(2r)\}, \{\sigma^3(2r), \sigma^4(2r)\}, \dots, \{\sigma^{4n-1}(2r), \sigma^{4n}(2r)\} = \{\sigma^{4n-1}(2r), 2r\}$ belongs to V' since $|V'| = 2n$. By Lemma2.2 $[2r, \sigma(2r)] \notin E(G), [\sigma^2(2r), \sigma^3(2r)] \notin E(G), \dots, [\sigma^{4n-2}(2r), \sigma^{4n-1}(2r)] \notin E(G)$ since $[\sigma^{4n-1}(2r), 2r] \in E(G)$ and $\sigma^{4n}(2r) = 2r$. Hence $2r \Rightarrow \sigma^2(2r) \Rightarrow \sigma^4(2r) \Rightarrow \dots \Rightarrow \sigma^{4n-2}(2r) \in V'$. Therefore the $2n$ vertices of C are $2r, \sigma^2(2r), \dots, \sigma^{4n-2}(2r)$ since the vertices $2r, \sigma^2(2r), \dots, \sigma^{4n-2}(2r)$ are mutually distinct. Also $\sigma^{2i}(2r)$ is even labelled where $1 \leq i \leq 2n - 1$. Hence $V' = \text{Even}(\sigma)$, since $|\text{Even}(\sigma)| = 2n$, thus a contradiction to the fact that $[2, 4] \notin E(G)$. Therefore V' does not have any even labelled vertex. Hence $V' = \text{Odd}(\sigma)$ when $[2, 4] \notin E(G)$. \square

Corollary 2.3 : *Let G be a s.c. graph with $p=4n$, $\omega(G) = 2n$ and a star c.p. $\sigma^* = (1\ 2 \dots 4n)$ where $n \geq 2$. Then $\langle \text{Even}(\sigma^*) \rangle$ is the unique maximum clique of G .*

Proof: Let C be a maximum clique of G with V' as its vertex set. By Lemma2.6 $V' = \text{Even}(\sigma^*)$ or $V' = \text{Odd}(\sigma^*)$. We note that $[2, 4] \in E$ since σ^* is a star c.p. . Hence by Lemma2.1 $[\sigma^*(2), \sigma^*(4)] = [3, 5] \notin E$. Therefore $V' \neq \text{Odd}(\sigma^*)$. \square

For a s.c. graph G with $4n$ vertices and a star c.p. σ^* the collection of all even labelled vertices of σ^* induces a maximum clique of G when $\omega(G) = 2n$.

Theorem 2.14 : *Let G be a s.c. graph with $p=4n$, $\omega(G) = 2n$ and a star c.p. σ^* . Then $\langle \text{Even}(\sigma^*) \rangle$ is a maximum clique of G .*

Proof: Let $p=4$. In this case $G \cong P_4$. By Theorem 2.1 $\sigma^* = (1\ 2\ 3\ 4)$ is the only star c.p. of G . Also $\langle \text{Even}(\sigma^*) \rangle$ is a maximum clique of G since $[2, 4] \in E(G)$. Hence the Theorem is true when $p=4$. Let $p \geq 8$. Let $\sigma^* = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $1 \leq i \leq s$. By Theorem 2.1 each p_i is divisible by 4. Let $V_i = \{v_{ij} : 1 \leq j \leq p_i\}$ where

$1 \leq i \leq s$. We note that $|V_i| = p_i$ where $1 \leq i \leq s$. Let C be a maximum clique of G and let V' be its vertex set. We prove that $|V' \cap V_i| = \frac{p_i}{2}$ for all $1 \leq i \leq s$. Let $|V' \cap V_j| \neq \frac{p_j}{2}$ for some $1 \leq j \leq s$. Then there exists a V_k such that $|V' \cap V_k| \geq \frac{p_k}{2} + 1$ for some $1 \leq k \leq s$. By Theorem 2.13 $\langle V_k \rangle$ is a s.c. graph. Hence by Theorem 2.11 every complete subgraph of $\langle V_k \rangle$ has at most $\frac{p_k}{2}$ vertices, thus a contradiction since $\langle V' \cap V_k \rangle$ is a complete subgraph of $\langle V_k \rangle$. Therefore $|V' \cap V_i| = \frac{p_i}{2}$ for all $1 \leq i \leq s$. By Corollary 2.3 C has all even labelled vertices of σ_i^* 's whose length is at least 8 since $\langle V_i \cap V' \rangle$ is a maximum clique of $\langle V_i \rangle$ where $1 \leq i \leq s$. We prove that C has both an odd labelled vertex and an even labelled vertex from at most one permutation cycle of length 4. Let C have both an even labelled and an odd labelled vertex from two distinct permutation cycles σ_m^* and σ_n^* each of length 4. Either $\sigma^*(v_{i2x}) = v_{i(2y+1)}$ or $\sigma^*(v_{i(2y+1)}) = v_{i2x}$ for any even labelled vertex v_{i2x} and for any odd labelled vertex $v_{i(2y+1)}$ of σ_i^* where σ_i^* is a permutation cycle of length 4 and $1 \leq i \leq s$. Therefore either of the cases (a), (b), (c) or (d) holds.

- (a) There exist even labelled vertices v_{l2x} and v_{m2y} of σ_l^* and σ_m^* respectively such that v_{l2x} , $\sigma^*(v_{l2x})$, v_{m2y} and $\sigma^*(v_{m2y})$ belong to V' .
- (b) There exist odd labelled vertices $v_{l(2x+1)}$ and $v_{m(2y+1)}$ of σ_l^* and σ_m^* respectively such that $v_{l(2x+1)}$, $\sigma^*(v_{l(2x+1)})$, $v_{m(2y+1)}$ and $\sigma^*(v_{m(2y+1)})$ belong to V' .
- (c) There exist an even labelled vertex v_{l2x} of σ_l^* and an odd labelled vertex $v_{m(2y+1)}$ of σ_m^* such that v_{l2x} , $\sigma^*(v_{l2x})$, $v_{m(2y+1)}$ and $\sigma^*(v_{m(2y+1)})$ belong to V' .
- (d) There exist an odd labelled vertex $v_{l(2x+1)}$ of σ_l^* and an even labelled vertex v_{m2y} of σ_m^* such that $v_{l(2x+1)}$, $\sigma^*(v_{l(2x+1)})$, v_{m2y} and $\sigma^*(v_{m2y})$ belong to V' .

All the cases (a), (b), (c) and (d) contradict Lemma 2.4. So C has both an even labelled vertex and an odd labelled vertex from at most one permutation cycle of length 4. We note that C cannot have the two odd labelled vertices of any permutation cycle of length 4 since σ^* is a star c.p. Hence C has only even labelled vertices of σ^* with a possible exception of a single odd labelled vertex of a permutation cycle σ_m^* of length 4 belonging to C . If C has only even labelled vertices of σ^* the Theorem is proved since C has $2n$ vertices. Otherwise an even labelled vertex v_{m2x} and an odd labelled vertex v_{m2y+1} of a permutation cycle σ_m^* of length

4 belong to C. Define $B = \{v_{ij} \in \text{Even}(\sigma_i^*) : 1 \leq i \leq s, i \neq m \text{ and } 1 \leq j \leq p_i\}$. We note that $\sigma^{*2}(B) = B$. Also $[v_{m2x}, B] \subseteq E(G)$ since v_{m2x} is a vertex in C not in B and $B \subseteq V'$. By Lemma2.2 $[\sigma^{*2}(v_{m2x}), \sigma^{*2}(B)] \subseteq E(G)$ since $[v_{m2x}, B] \subseteq E(G)$. So $[\sigma^{*2}(v_{m2x}), B] \subseteq E(G)$ since $\sigma^{*2}(B) = B$. The permutation cycle σ_m^* is of length 4 implies $\sigma^{*2}(v_{m2x})$ is an even labelled vertex distinct from v_{m2x} . The vertices v_{m2x} and $\sigma^{*2}(v_{m2x})$ are the only even labelled vertices of the permutation cycle σ_m^* implies $[v_{m2x}, \sigma^{*2}(v_{m2x})] \in E(G)$ since σ^* is a star c.p. of G. We have $\text{Even}(\sigma^*) = B \cup \{v_{m2x}, \sigma^{*2}(v_{m2x})\}$. Therefore $\langle \text{Even}(\sigma^*) \rangle$ is a maximum clique of G since $[v_{m2x}, B] \subseteq E(G)$, $[\sigma^{*2}(v_{m2x}), B] \subseteq E(G)$, $[v_{m2x}, \sigma^{*2}(v_{m2x})] \in E(G)$, B is contained in the vertex set of C and $|\text{Even}(\sigma^*)| = 2n$. \square

The following result is a sufficient condition for a s.c. graph with $4n$ vertices to be a split graph in terms of its clique number.

Theorem 2.15 : *Let G be a s.c. graph with $p=4n$ and $\omega(G) = 2n$. Then G is a split graph.*

Proof: Let σ^* be a star c.p. of G. By Theorem 2.14 $\langle \text{Even}(\sigma^*) \rangle$ is a maximum clique of G. We note $\text{Odd}(\sigma^*)$ is a stable set of G since $\text{Odd}(\sigma^*) = \sigma^*(\text{Even}(\sigma^*))$. Also the sets $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$ partition $V(G)$. Hence G is a split graph. \square

We obtain a sufficient condition for a s.c. graph with $4n+1$ vertices to be a split graph in terms of its clique number.

Theorem 2.16 : *Let G be a s.c. graph with $p=4n+1$ and $\omega(G) = 2n+1$. Then G is a split graph.*

Proof: Let σ be a c.p. of G with v_0 as its fixed vertex. Let $G' = G - v_0$. By Corollary2.2 G' is a s.c. graph. Let C be a maximum clique of G. Let V_C be the vertex set of C. We prove that $v_0 \in V_C$. Let $v_0 \notin V_C$. We note that C is a maximum clique of G' since $v_0 \notin V_C$. Hence $\omega(G') = 2n+1$, a contradiction to Theorem2.11. So $v_0 \in V_C$. Let $S = \sigma(V_C)$. We note that $\sigma(v_0) = v_0$ since v_0 is the fixed vertex of σ . Hence $v_0 \in S$ since $v_0 \in V_C$. Also S is a stable set of G since $\langle V_C \rangle$ is a clique of G. We note that S and V_C have atmost one common element since $\langle V_C \rangle$ is a clique of G and S is a stable set of G. Also $|S| = |V_C| = 2n+1$. Hence the sets V_C and $S - \{v_0\}$ partition $V(G)$. Therefore G is a split graph. \square

The following result is a consolidation of Theorem2.15 and Theorem2.16.

Theorem 2.17 : *Let G be a s.c. graph. Then G is a split graph if*

- (i) $\omega(G) = 2n$ when $p=4n$
- (ii) $\omega(G) = 2n + 1$ when $p=4n+1$.

Proof: Follows from Theorem 2.15 and Theorem 2.16. \square

A characterisation of a s.c. graph to be a split graph in terms of its clique number follows.

Theorem 2.18 : *Let G be a s.c. graph. Then G is a split graph if and only if*

- (i) $\omega(G) = 2n$ when $p=4n$
- (ii) $\omega(G) = 2n + 1$ when $p=4n+1$.

Proof: Follows from Theorem 2.10 and Theorem 2.17. \square

Theorem 2.19 : *Let G be a s.c. graph. Then G is a split graph if and only if G is a chordal graph.*

Proof: Follows from Theorem 2.3. \square

From the above result we see that the class of s.c. split graphs is exactly the class of s.c. chordal graphs. Hence we have the following characterisation for s.c. graphs to be chordal in terms of its clique number.

Theorem 2.20 : *Let G be a s.c. graph. Then G is a chordal graph if and only if*

- (i) $\omega(G) = 2n$ when $p=4n$
- (ii) $\omega(G) = 2n + 1$ when $p=4n+1$.

Proof: Follows from Theorem 2.18 and Theorem 2.19. \square

2.4 A characterisation for self-complementary graphs to be chordal in terms of the stability number

The clique number of a graph is equal to the stability number of its complement.

Theorem 2.21 : *Let G be a graph. Then $\omega(G) = \alpha(\bar{G})$.*

Proof: Let $\omega(G) > \alpha(\bar{G})$. Let C be a maximum clique of G . Let V' be the vertex set of C . We note that V' is a stable set of \bar{G} since C is a clique of G . So $\alpha(\bar{G}) \geq \omega(G)$ since $|V'| = \omega(G)$, a contradiction. Then $\omega(G) \leq \alpha(\bar{G})$. Let $\omega(G) < \alpha(\bar{G})$. Let S be a maximum stable set of \bar{G} . The set S induces a complete subgraph in G . Hence $\alpha(\bar{G}) \leq \omega(G)$ since $|S| = \alpha(\bar{G})$, a contradiction. Therefore $\omega(G) = \alpha(\bar{G})$. \square

For a s.c. graph the clique number is equal to its stability number.

Theorem 2.22 : *Let G be a s.c. graph. Then $\omega(G) = \alpha(G)$.*

Proof: Follows from Theorem 2.21 since G is a s.c. graph. \square

The following result is a characterisation for a s.c. graph to be a chordal graph in terms of its stability number.

Theorem 2.23 : *Let G be a s.c. graph. Then G is a chordal graph if and only if*

- (i) $\alpha(G) = 2n$ when $p=4n$
- (ii) $\alpha(G) = 2n + 1$ when $p=4n+1$.

Proof: Follows from Theorem 2.20 and Theorem 2.22. \square

2.5 A characterisation for self-complementary graphs to be chordal in terms of the induced cycles

Theorem 2.24 *Let G be a s.c. graph. Then G has no induced subgraph isomorphic to C_4 if and only if it has no induced subgraph isomorphic to $2K_2$.*

Proof: By Lemma 2.3 G has no induced subgraph isomorphic to C_4 if and only if it has no induced subgraph isomorphic to $\bar{C}_4 = 2K_2$. \square

Lemma 2.7 : *Let G be a s.c. graph with $p=4n$ and a c.p. $\sigma = (1\ 2\ \dots\ 4n)$. Then*

$[2, 2k + 2] \in E(G)$ if and only if $[2, 4n + 2 - 2k] \in E(G)$ where k is an integer such that $1 \leq k \leq n$.

Proof: By Lemma2.2 $[2, 2k + 2] \in E(G)$ if and only if $[\sigma^{4n-2k}(2), \sigma^{4n-2k}(2k + 2)] = [4n + 2 - 2k, 2] \in E(G)$ where $1 \leq k \leq n$. \square

Lemma 2.8 : *Let G be a s.c. graph with $p=4n$ and a c.p. $\sigma = (1\ 2 \cdots 4n)$. Then $[2, 2k + 2] \in E(G)$ for all integers k such that $n \leq k \leq 2n - 1$ if and only if $[2, 2k + 2] \in E(G)$ for all integers k such that $1 \leq k \leq n$.*

Proof: Every vertex $2k + 2$ where $n \leq k \leq 2n - 1$ can be represented as $4n + 2 - 2k$ for some integer k such that $1 \leq k \leq n$ and every vertex $4n + 2 - 2k$ where $1 \leq k \leq n$ can be represented as $2k + 2$ for some integer k such that $n \leq k \leq 2n - 1$. So $[2, 2k + 2] \in E(G)$ for all $n \leq k \leq 2n - 1$ if and only if $[2, 4n + 2 - 2k] \in E(G)$ for all $1 \leq k \leq n$. By Lemma2.7 $[2, 4n + 2 - 2k] \in E(G)$ for all $1 \leq k \leq n$ if and only if $[2, 2k + 2] \in E(G)$ for all $1 \leq k \leq n$. Hence $[2, 2k + 2] \in E(G)$ for all $n \leq k \leq 2n - 1$ if and only if $[2, 2k + 2] \in E(G)$ for all $1 \leq k \leq n$. \square

Lemma 2.9 : *Let G be a s.c. graph with $p=4n$ and a c.p. $\sigma = (1\ 2 \cdots 4n)$. Let j be an integer such that $1 \leq j \leq 2n - 1$. Then $[2, 2k + 2] \in E(G)$ for all integers k such that $1 \leq k \leq 2n - 1$ if and only if $[2j + 2, \sigma^{2k}(2j + 2)] \in E(G)$ for all integers k such that $1 \leq k \leq 2n - 1$.*

Proof: By Lemma2.2 $[2, 2k + 2] \in E(G)$ for all $1 \leq k \leq 2n - 1$ if and only if $[\sigma^{2j}(2), \sigma^{2j}(2k + 2)] = [2j + 2, \sigma^{2k}(2j + 2)] \in E(G)$ for all $1 \leq k \leq 2n - 1$. \square

Lemma 2.10 : *Let G be a s.c. graph with $p=8n$ and a c.p. $\sigma = (1\ 2 \cdots 8n)$. Then $[2, 4n + 2] \in E(G)$ and $[2, 2n + 2] \notin E(G)$ implies G has an induced C_4 .*

Proof: Let $[2, 4n + 2] \in E(G)$ and $[2, 2n + 2] \notin E(G)$. By Lemma2.2 $[\sigma^{2n}(2), \sigma^{2n}(4n + 2)] = [2n + 2, 6n + 2] \in E(G)$ since $[2, 4n + 2] \in E(G)$. Also by Lemma2.2 $[\sigma^{2n}(2), \sigma^{2n}(2n + 2)] = [2n + 2, 4n + 2] \notin E(G)$, $[\sigma^{4n}(2), \sigma^{4n}(2n + 2)] = [4n + 2, 6n + 2] \notin E(G)$ and $[\sigma^{6n}(2), \sigma^{6n}(2n + 2)] = [6n + 2, 2] \notin E(G)$ since $[2, 2n + 2] \notin E(G)$. Hence the vertices $2, 2n+2, 4n+2$ and $6n+2$ induce a $2K_2$. Then by Theorem2.24 G has an induced C_4 . \square

Lemma 2.11 : *Let G be a s.c. graph with $p=4n$ and a star c.p. $\sigma^* = (1\ 2 \cdots 4n)$. Then $[2, 2n + 2] \notin E(G)$ implies that G has an induced C_4 .*

Proof: Let $[2, 2n+2] \notin E(G)$. We note that $[2, 4] \in E(G)$ since σ^* is a star c.p.. By Lemma2.2 $[\sigma^{*2n}(2), \sigma^{*2n}(4)] = [2n+2, 2n+4] \in E(G)$ since $[2, 4] \in E(G)$. Also by Lemma2.2 $[\sigma^{*2}(2), \sigma^{*2}(2n+2)] = [4, 2n+4] \notin E(G)$ since $[2, 2n+2] \notin E(G)$.

Case i : Let $[4, 2n+2] \notin E(G)$. By Lemma2.2 $[\sigma^{*2n}(4), \sigma^{*2n}(2n+2)] = [2n+4, 2] \notin E(G)$ since $[4, 2n+2] \notin E(G)$. Hence the vertices 2, 4, 2n+2 and 2n+4 induce a $2K_2$. Then by Theorem2.24 G has an induced C_4 .

Case ii : Let $[4, 2n+2] \in E(G)$. By Lemma2.2 $[\sigma^{*2n}(4), \sigma^{*2n}(2n+2)] = [2n+4, 2] \in E(G)$ since $[4, 2n+2] \in E(G)$. Hence the vertices 2, 4, 2n+2 and 2n+4 induce a C_4 .

Therefore the Lemma. \square

Lemma 2.12 : Let G be a s.c. graph with $p=4n$ and a star c.p. $\sigma^* = (1\ 2 \cdots 4n)$. Then G is C_4 -free implies that $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G.

Proof: Let G be C_4 -free. By Lemma2.9 for proving $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G, we have to prove that $[2, 2i+2] \in E(G)$ for all $1 \leq i \leq 2n-1$. By Lemma2.8 for proving $[2, 2i+2] \in E(G)$ for all $n \leq i \leq 2n-1$ we have to prove that $[2, 2i+2] \in E(G)$ for all $1 \leq i \leq n$. Hence for proving $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G we prove that $[2, 2i+2] \in E(G)$ for all $1 \leq i \leq n$. First we note that $[2, 2n+2] \in E(G)$. Otherwise by Lemma2.11 G has an induced C_4 , a contradiction to the assumption that G is C_4 -free. Now we prove that $[2, 2i+2] \in E(G)$ for all $1 \leq i \leq n-1$. Let $[2, 2k+2] \notin E(G)$ for an integer k such that $1 \leq k \leq n-1$. Consider the vertex $2n+2-2k$. Either $2n+2-2k = 2k+2$ or $2n+2-2k \neq 2k+2$. Let $2n+2-2k = 2k+2$. That is $n = 2k$. Then G is a s.c. graph with $p = 8k$ and a star c.p. $\sigma^* = (1\ 2 \cdots 8k)$. Also $[2, 4k+2] \in E(G)$ since $[2, 2n+2] \in E(G)$. By Lemma2.10 G has an induced C_4 since $[2, 2k+2] \notin E(G)$, a contradiction to the assumption that G is C_4 -free. Therefore $2n+2-2k \neq 2k+2$. That is the vertices $2k+2$ and $2n+2-2k$ are distinct. By Lemma2.2 $[\sigma^{*2k}(2), \sigma^{*2k}(2n+2)] = [2k+2, 2n+2k+2] \in E(G)$ since $[2, 2n+2] \in E(G)$. Also by Lemma2.2 $[\sigma^{*2n}(2), \sigma^{*2n}(2k+2)] = [2n+2, 2n+2k+2] \notin E(G)$ since $[2, 2k+2] \notin E(G)$.

Case i : Let $[2, 2n+2-2k] \notin E(G)$. By Lemma2.2 $[\sigma^{*2k}(2), \sigma^{*2k}(2n+2-2k)] = [2k+2, 2n+2] \notin E(G)$ and $[\sigma^{*2n+2k}(2), \sigma^{*2n+2k}(2n+2-2k)] = [2n+2k+2, 2] \notin E(G)$ since $[2, 2n+2-2k] \notin E(G)$. Hence the vertices 2, 2k+2, 2n+2 and 2n+2k+2 induce a $2K_2$. Then

by Theorem 2.24 G has an induced C_4 , a contradiction to the assumption that G is C_4 -free.

Case ii : Let $[2, 2n + 2 - 2k] \in E(G)$. By Lemma 2.2 $[\sigma^{*2k}(2), \sigma^{*2k}(2n + 2 - 2k)] = [2k + 2, 2n + 2] \in E(G)$ and $[\sigma^{*2n+2k}(2), \sigma^{*2n+2k}(2n + 2 - 2k)] = [2n + 2k + 2, 2] \in E(G)$ since $[2, 2n + 2 - 2k] \in E(G)$. Hence the vertices 2, $2k+2$, $2n+2$ and $2n+2k+2$ induce a C_4 , a contradiction to the assumption that G is C_4 -free.

Therefore $[2, 2i + 2] \in E(G)$ for all $1 \leq i \leq n - 1$. \square

Lemma 2.13 : Let G be a s.c. graph with $p=4n$ and a c.p. $\sigma = \sigma_1\sigma_2 \cdots \sigma_s$ where

$\sigma_i = (v_{i1}v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$. Then $[Even(\sigma_i), Even(\sigma_j)] \subseteq E(G)$ for all integers i and j such that $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$ if and only if $[v_{i2}, Even(\sigma_j)] \subseteq E(G)$ for all integers i and j such that $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$.

Proof: By Theorem 2.1 p_i is divisible by 4 for all $1 \leq i \leq s$. We note that $[v_{i2k}, Even(\sigma_j)] = [\sigma^{2k-2}(v_{i2}), \sigma^{2k-2}(Even(\sigma_j))] \subseteq E$ for all $1 \leq k \leq \frac{p_i}{2}$, $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$ if and only if $[v_{i2}, Even(\sigma_j)] \subseteq E$ for all $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$. \square

For a s.c. graph G with $4n$ vertices and a star c.p. σ^* the collection of all even labelled vertices of σ^* induces a complete subgraph of G when G is C_4 -free.

Theorem 2.25 : Let G be a s.c. graph with $p=4n$ and a star c.p. σ^* . Then G is C_4 -free implies $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G .

Proof: Let G be C_4 -free. Let $\sigma^* = \sigma_1^*\sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1}v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$. Let $V_i = \{v_{ij} : 1 \leq j \leq p_i\}$ for all $1 \leq i \leq s$. By Theorem 2.1 p_i is divisible by 4 for all $1 \leq i \leq s$. By Theorem 2.13 $\langle V_i \rangle$ is a s.c. graph with a star c.p. $(v_{i1}v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$. Also $\langle V_i \rangle$ is C_4 -free for all $1 \leq i \leq s$ since G is C_4 -free. Hence by Lemma 2.12 $\langle Even(\sigma_i^*) \rangle$ is a complete subgraph of $\langle V_i \rangle$ where $1 \leq i \leq s$ and hence a complete subgraph of G . Then to prove that $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G we have to prove that $[Even(\sigma_i^*), Even(\sigma_j^*)] \subseteq E(G)$ for all $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$. By Lemma 2.13 to prove $[Even(\sigma_i^*), Even(\sigma_j^*)] \subseteq E(G)$ for all $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$, we have to prove that $[v_{i2}, Even(\sigma_j^*)] \subseteq E(G)$ for all $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$. Therefore to prove $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G we prove that $[v_{i2}, Even(\sigma_j^*)] \subseteq E(G)$ for all $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$. Let $[v_{i2}, Even(\sigma_k^*)] \not\subseteq E(G)$ for two distinct integers

l and k such that $1 \leq l \leq s$ and $1 \leq k \leq s$. There exists a vertex v_{k2x} of $Even(\sigma_k^*)$ such that $[v_{l2}, v_{k2x}] \notin E(G)$ since $[v_{l2}, Even(\sigma_k^*)] \not\subseteq E(G)$. We prove that there exists a vertex v_{k2y} of $Even(\sigma_k^*)$ such that $[v_{l2}, v_{k2y}] \in E(G)$. Otherwise $[v_{l2}, v_{k2i}] \notin E(G)$ for all $v_{k2i} \in Even(\sigma_k^*)$. We note that $[v_{l2}, v_{k2i}] \notin E(G)$ for all $v_{k2i} \in Even(\sigma_k^*)$ implies $[v_{l2}, v_{k2}] \notin E(G)$ and $[v_{l2}, v_{k4}] \notin E(G)$. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2}(v_{k2i})] = [v_{l4}, \sigma^{*2}(v_{k2i})] \notin E(G)$ for all $v_{k2i} \in Even(\sigma_k^*)$ since $[v_{l2}, v_{k2i}] \notin E(G)$ for all $v_{k2i} \in Even(\sigma_k^*)$. Then $[v_{l4}, v_{k2}] \notin E(G)$ and $[v_{l4}, v_{k4}] \notin E(G)$. We have $[v_{l2}, v_{l4}] \in E(G)$ and $[v_{k2}, v_{k4}] \in E(G)$ since $\langle Even(\sigma_l^*) \rangle$ and $\langle Even(\sigma_k^*) \rangle$ are complete subgraphs of G . Therefore the vertices v_{l2}, v_{l4}, v_{k2} and v_{k4} induces a $2K_2$. Then by Theorem2.24 G has an induced C_4 , a contradiction to the assumption that G is C_4 -free. Hence there exists a vertex v_{k2y} of $Even(\sigma_k^*)$ such that $[v_{l2}, v_{k2y}] \in E(G)$. We note that one of the following cases should hold.

- (a) $[v_{l2}, v_{k2}] \in E(G)$ and $[v_{l4}, v_{k2}] \notin E(G)$
- (b) $[v_{l2}, v_{k2}] \notin E(G)$ and $[v_{l4}, v_{k2}] \in E(G)$
- (c) $[v_{l2}, v_{k2}] \in E(G)$ and $[v_{l4}, v_{k2}] \in E(G)$
- (d) $[v_{l2}, v_{k2}] \notin E(G)$ and $[v_{l4}, v_{k2}] \notin E(G)$.

We prove that the cases (a), (b), (c) and (d) are impossible.

(a) Choose the vertex v_{k2m} of $Even(\sigma_k^*)$ such that $[v_{l2}, v_{k2m}] \notin E(G)$ and $[v_{l2}, v_{k2i}] \in E(G)$ for all $1 \leq i \leq m-1$ since $[v_{l2}, v_{k2x}] \notin E(G)$ the vertex v_{k2m} exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2}(v_{k2(m-1)})] = [v_{l4}, v_{k2m}] \in E(G)$ since $[v_{l2}, v_{k2(m-1)}] \in E(G)$. Also $[v_{l2}, v_{l4}] \in E(G)$ and $[v_{k2}, v_{k2m}] \in E(G)$ since $\langle Even(\sigma_l^*) \rangle$ and $\langle Even(\sigma_k^*) \rangle$ are complete subgraphs of G . Hence the vertices v_{l2}, v_{l4}, v_{k2} and v_{k2m} induce a C_4 , a contradiction to the assumption that G is C_4 -free. Therefore this case is not possible.

(b) Choose the vertex v_{k2m} of $Even(\sigma_k^*)$ such that $[v_{l2}, v_{k2m}] \in E(G)$ and $[v_{l2}, v_{k2i}] \notin E(G)$ for all $1 \leq i \leq m-1$, since $[v_{l2}, v_{k2y}] \in E(G)$ the vertex v_{k2m} exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2}(v_{k2(m-1)})] = [v_{l4}, v_{k2m}] \notin E(G)$ since $[v_{l2}, v_{k2(m-1)}] \notin E(G)$. Also $[v_{l2}, v_{l4}] \in E(G)$ and $[v_{k2}, v_{k2m}] \in E(G)$ since $\langle Even(\sigma_l^*) \rangle$ and $\langle Even(\sigma_k^*) \rangle$ are complete subgraphs of G . Hence the vertices v_{l2}, v_{l4}, v_{k2} and v_{k2m} induce a C_4 , a contradiction to the assumption that G is C_4 -free. Therefore this case is not possible.

(c) Choose the vertex $v_{k2m} \in \text{Even}(\sigma_k^*)$ such that $[v_{l2}, v_{k2m}] \notin E(G)$ and $[v_{l2}, v_{k2i}] \in E(G)$ for all $1 \leq i \leq m-1$ since $[v_{l2}, v_{k2x}] \notin E(G)$ the vertex v_{k2m} exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2}(v_{k2(m-1)})] = [v_{l4}, v_{k2m}] \in E(G)$ since $[v_{l2}, v_{k2(m-1)}] \in E(G)$. Now choose the vertex $\sigma^{*2r}(v_{k2m}) \in \text{Even}(\sigma_k^*)$ such that $[v_{l2}, \sigma^{*2r}(v_{k2m})] \in E(G)$ and $[v_{l2}, \sigma^{*2i}(v_{k2m})] \notin E(G)$ for all $0 \leq i \leq r-1$, since $[v_{l2}, v_{k2}] \in E(G)$ the vertex $\sigma^{*2r}(v_{k2m})$ exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2r}(v_{k2m})] = [v_{l4}, \sigma^{*2r}(v_{k2m})] \notin E(G)$ since $[v_{l2}, \sigma^{*2(r-1)}(v_{k2m})] \notin E(G)$. Also $[v_{l2}, v_{l4}] \in E(G)$ and $[v_{k2m}, \sigma^{*2r}(v_{k2m})] \in E(G)$ since $\langle \text{Even}(\sigma_l^*) \rangle$ and $\langle \text{Even}(\sigma_k^*) \rangle$ are complete subgraphs of G . Hence the vertices v_{l2}, v_{l4}, v_{k2m} and $\sigma^{*2r}(v_{k2m})$ induce a C_4 , a contradiction to the assumption that G is C_4 -free. Therefore this case is not possible.

(d) Choose the vertex $v_{k2m} \in \text{Even}(\sigma_k^*)$ such that $[v_{l2}, v_{k2m}] \in E(G)$ and $[v_{l2}, v_{k2i}] \notin E(G)$ for all $1 \leq i \leq m-1$ since $[v_{l2}, v_{k2y}] \in E(G)$ the vertex v_{k2m} exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2}(v_{k2(m-1)})] = [v_{l4}, v_{k2m}] \notin E(G)$ since $[v_{l2}, v_{k2(m-1)}] \notin E(G)$. Now choose the vertex $\sigma^{*2r}(v_{k2m}) \in \text{Even}(\sigma_k^*)$ such that $[v_{l2}, \sigma^{*2r}(v_{k2m})] \notin E(G)$ and $[v_{l2}, \sigma^{*2i}(v_{k2m})] \in E(G)$ for all $0 \leq i \leq r-1$, since $[v_{l2}, v_{k2}] \notin E(G)$ the vertex $\sigma^{*2r}(v_{k2m})$ exists. By Lemma2.2 $[\sigma^{*2}(v_{l2}), \sigma^{*2r}(v_{k2m})] = [v_{l4}, \sigma^{*2r}(v_{k2m})] \in E(G)$ since $[v_{l2}, \sigma^{*2(r-1)}(v_{k2m})] \in E(G)$. Also $[v_{l2}, v_{l4}] \in E(G)$ and $[v_{k2m}, \sigma^{*2r}(v_{k2m})] \in E(G)$ since $\langle \text{Even}(\sigma_l^*) \rangle$ and $\langle \text{Even}(\sigma_k^*) \rangle$ are complete subgraphs of G . Hence the vertices v_{l2}, v_{l4}, v_{k2m} and $\sigma^{*2r}(v_{k2m})$ induce a C_4 , a contradiction to the assumption that G is C_4 -free. Therefore this case is not possible.

So the cases (a), (b), (c) and (d) are impossible, a contradiction to the fact that either one of them should hold. Therefore $[v_{i2}, \text{Even}(\sigma_j^*)] \subseteq E(G)$ for $1 \leq i \leq s$, $1 \leq j \leq s$ and $i \neq j$. \square

Every C_4 -free s.c. graph with $4n$ vertices is a split graph.

Theorem 2.26 : Let G be a s.c. graph with $p=4n$. Then G is C_4 -free implies that G is a split graph.

Proof: Let G be C_4 -free. Let σ^* be a star c.p. of G . By Theorem2.25 $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . We note that $\text{Odd}(\sigma^*)$ is a stable set of G since $\text{Odd}(\sigma^*) = \sigma^*(\text{Even}(\sigma^*))$. Hence G is a split graph since the sets $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$ partition $V(G)$. \square

Theorem 2.27 : *Let G be a s.c. graph with $p=4n$. Then G is split graph implies that G has no induced subgraph isomorphic to C_5 .*

Proof: Follows from Theorem2.2. \square

Every s.c. graph with $4n$ vertices having an induced subgraph isomorphic to C_5 has an induced subgraph isomorphic to C_4 .

Theorem 2.28 : *Let G be a s.c. graph with $p=4n$. Then G has no induced subgraph isomorphic to C_4 implies that G has no induced subgraph isomorphic to C_5 .*

Proof: Let G be a s.c. graph with $4n$ vertices which has no induced subgraph isomorphic to C_4 and has an induced subgraph isomorphic to C_5 . By Theorem2.26 G is a split graph since G is C_4 -free. Then by Theorem2.27 G has no induced subgraph isomorphic to C_5 since G is a split graph, a contradiction. \square

However, in the case of a s.c. graph G with $4n+1$ vertices G has no induced subgraph isomorphic to C_4 is not sufficient to deduce that G has no induced subgraph isomorphic to C_5 . For example the s.c. graph C_5 has no induced subgraph isomorphic to C_4 .

Theorem 2.29 : *Let G be a s.c. graph with $p=4n$. Then G has no induced subgraph isomorphic to C_4 implies that G has no induced subgraph isomorphic to $2K_2$ or C_5 .*

Proof: Follows from Theorem2.24 and Theorem2.28. \square

The following result is a characterisation for a s.c. graph to be a split graph in terms of its induced cycles.

Theorem 2.30 : *Let G be a s.c. graph. Then G is a split graph if and only if G has no induced subgraph isomorphic to C_4 or C_5 .*

Proof: Follows from Theorem2.2 and Theorem2.24. \square

We have the following characterisation for s.c. graphs to be split graphs in terms of its induced cycles which is a modification of Theorem2.30 for the case of s.c. graph with $4n$ vertices.

Theorem 2.31 : *Let G be a s.c. graph. Then G is a split graph if and only if*

- (i) G has no induced subgraph isomorphic to C_4 when $p=4n$
- (ii) G has no induced subgraph isomorphic to C_4 or C_5 when $p=4n+1$.

Proof: (i) Follows from Theorem2.2 and Theorem2.29.

(ii) Follows from Theorem2.30. \square

The following result is a characterisation for a s.c. graph to be a chordal graph in terms of its induced cycles.

Theorem 2.32 : *Let G be a s.c. graph. Then G is a chordal graph if and only if*

- (i) G has no induced subgraph isomorphic to C_4 when $p=4n$
- (ii) G has no induced subgraph isomorphic to C_4 or C_5 when $p=4n+1$.

Proof: Follows from Theorem2.19 and Theorem2.31. \square

2.6 A characterisation for self-complementary graphs to be chordal in terms of the degree sequence

Hammer and Simone [109] obtained a characterisation for split graphs in terms of its degree sequence. We obtain a characterisation for s.c. graphs to be split graphs in terms of its degree sequence.

Theorem 2.33 *Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then G is a split graph if and only if*

- (i) $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ when $p=4n$
- (ii) $\sum_{i=1}^{2n} d_i = 6n^2$ when $p=4n+1$.

Proof: (i) Let $p=4n$. Let $\mathcal{M} = \max\{i : d_i \geq i - 1, 1 \leq i \leq p\}$. By Theorem2.9 $\mathcal{M} = 2n$. By Theorem2.8 $d_i = 4n - 1 - d_{4n-i+1}$ for all $2n + 1 \leq i \leq 4n$. Then $\sum_{i=2n+1}^{4n} d_i = 8n^2 - 2n - \sum_{i=1}^{2n} d_i$. Hence by Theorem2.5 the graph G is a split graph if and only if $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$.

(ii) Let $p=4n+1$. Let $\mathcal{M} = \max\{i : d_i \geq i-1, 1 \leq i \leq p\}$. By Theorem 2.9 $\mathcal{M} = 2n+1$. By Theorem 2.8 $d_i = 4n - d_{4n-i+2}$ for all $2n+2 \leq i \leq 4n+1$. Then $\sum_{i=2n+2}^{4n+1} d_i = 8n^2 - \sum_{i=1}^{2n} d_i$. Also by Theorem 2.8 $d_{2n+1} = 2n$. Hence by Theorem 2.5 the graph G is a split graph if and only if $\sum_{i=1}^{2n} d_i = 6n^2$. \square

A characterisation for a s.c. graph to be a chordal graph in terms of its degree sequence follows.

Theorem 2.34 *Let G be a s.c. graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$. Then G is a chordal graph if and only if*

(i) $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ when $p=4n$

(ii) $\sum_{i=1}^{2n} d_i = 6n^2$ when $p=4n+1$.

Proof: Follows from Theorem 2.19 and Theorem 2.33. \square

Chapter 3

On the existence and the construction of self-complementary chordal graphs

3.1 Introduction

Existence problems of self-complementary graphs had been studied by Ringel[178] and Sachs[190]. They prove that a s.c. graph with p vertices exists if and only if $p=4n$ or $p=4n+1$ for some positive integer n . In Section 3.3 we prove that a s.c. chordal graph with p vertices exists if and only if $p=4n$ or $p=4n+1$ for some positive integer n .

Gibbs[98] has given algorithms for the construction of all s.c. graphs with $4n$ vertices and $4n+1$ vertices. We discuss these algorithms in Section 3.2. In Section 3.4 by modifying these algorithms we give algorithms for the construction of all s.c. chordal graphs with $4n$ vertices and $4n+1$ vertices.

3.2 Algorithms for constructing self-complementary graphs

The following algorithm was given by Gibbs [98] for constructing all s.c. graphs with $4n$ vertices.

Algorithm 3.1 (Gibbs [98]) : An algorithm to construct s.c. graphs with $4n$ vertices where n is a positive integer

Choose positive integers p_1, p_2, \dots, p_s such that $\sum_{i=1}^s p_i = 4n$, $p_i \leq p_j$ for all $1 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $1 \leq i \leq s$. Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_s$, where $\sigma_i = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $1 \leq i \leq s$ be a permutation on $4n$ labels $v_{11}, v_{12}, \dots, v_{1p_1}, v_{21}, v_{22}, \dots, v_{2p_2}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. For the label v_{i1} where $1 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, v_{(i+2)1}, v_{(i+2)2}, \dots, v_{(i+2)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_i}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. Construct a graph G with $4n$ vertices by identifying the vertices of G with the labels v_{ij} where $1 \leq i \leq s$ and $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_{i1}, v_{jr}]$ where $1 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$.

(b) After completing (a) whether the pairs $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ or $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ such that $1 \leq i \leq s$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and the pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The following Theorem states that a graph G is a s.c. graph with $4n$ vertices if and only if it can be constructed by Algorithm3.1.

Theorem 3.1 (Gibbs [98]) : (i) The graphs constructed by Algorithm3.1 are s.c. graphs with $4n$ vertices where n is a positive integer.

(ii) Every s.c. graph with $4n$ vertices for all positive integer n can be constructed by Algorithm3.1.

We illustrate Algorithm3.1 by constructing a s.c. graph with 12 vertices. Two positive integers $p_1 = 4$ and $p_2 = 8$ are chosen. We note that the positive integers p_1 and p_2 are such that $p_1 + p_2 = 12$, $p_1 \leq p_2$, $p_1 = 2^2$ and $p_2 = 2^3$. Let $\sigma = (v_{11} v_{12} v_{13} v_{14})(v_{21} v_{22} v_{23} v_{24} v_{25} v_{26} v_{27} v_{28})$ be a permutation on 12 labels v_{ij} where $i = 1, 2$ and $1 \leq j \leq p_i$. For the label v_{11} the labels $v_{12}, v_{13}, v_{21}, v_{22}, v_{23}$ and v_{24} constitute its range. For the label v_{21} the labels v_{22}, v_{23}, v_{24} and v_{25} constitute its range. We construct a graph G with 12 vertices by identifying the vertices of G with the labels v_{ij} where $i = 1, 2$ and $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_{i1}, v_{jr}]$ where $i = 1, 2$ and v_{jr} is a vertex in the range of v_{i1} we decide arbitrarily whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$. Let the vertex v_{11} be adjacent to the vertices v_{22}, v_{23} and v_{24} and non-adjacent to the vertices v_{12}, v_{13} and v_{21} of its range as shown in Figure 3.1(a). Let the vertex v_{21} be adjacent to the vertices v_{22}, v_{23} and v_{25} and non-adjacent to the vertex v_{24} of its range as shown in Figure 3.1(a).

(b) After completing (a) whether $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ or $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ where $i = 1, 2$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The labelled graph G constructed is shown in Figure 3.1(b). We note that σ is a c.p. of the s.c. graph G .

The following algorithm was given by Gibbs [98] for the construction of all s.c. graphs with $4n+1$ vertices.

Algorithm 3.2 (Gibbs [98]) : An algorithm to construct s.c. graphs with $4n+1$ vertices where n is a positive integer

Choose positive integers p_2, p_3, \dots, p_s such that $\sum_{i=2}^s p_i = 4n$, $p_i \leq p_j$ for all

$2 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $2 \leq i \leq s$. Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_s$ where $\sigma_1 = (v_0)$ and $\sigma_i = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $2 \leq i \leq s$ be a permutation on $4n+1$ labels $v_0, v_{21}, v_{22}, \dots, v_{2p_2}, v_{31}, v_{32}, \dots, v_{3p_3}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. For the label v_0 define $v_{21}, v_{31}, \dots, v_{s1}$ as its range. For the label v_{i1} where $2 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, v_{(i+2)1}, v_{(i+2)2}, \dots, v_{(i+2)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_i}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. Construct a graph G with $4n+1$ vertices by identifying the vertices of G with the labels v_0 and v_{ij} where $2 \leq i \leq s$, $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_0, v_{i1}]$ where v_{i1} is in the range of v_0 it is arbitrarily decided whether $[v_0, v_{i1}] \in E(G)$ or $[v_0, v_{i1}] \notin E(G)$.

(b) After completing (a) whether the pairs $[\sigma^k(v_0), \sigma^k(v_{i1})] \in E(G)$ or $[\sigma^k(v_0), \sigma^k(v_{i1})] \notin E(G)$ for all $1 \leq k \leq p_i - 1$ is determined as follows for each pair $[v_0, v_{i1}]$ such that v_{i1} is in the range of v_0 . The pair $[\sigma^k(v_0), \sigma^k(v_{i1})] \in E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is

even and $[\sigma^k(v_0), \sigma^k(v_{i1})] \notin E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is odd.

(c) For each pair $[v_{i1}, v_{jr}]$ where $2 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$.

(d) After completing (c) whether the pairs $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ or $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ such that $2 \leq i \leq s$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and the pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The following Theorem states that a graph G is a s.c. graph with $4n+1$ vertices if and only if it can be constructed by Algorithm 3.2.

Theorem 3.2 (Gibbs [98]) : (i) The graphs constructed by Algorithm 3.2 are s.c. graphs with $4n+1$ vertices where n is a positive integer.

(ii) Every s.c. graph with $4n+1$ vertices for all positive integer n can be constructed by Algorithm 3.2.

We illustrate Algorithm 3.2 by constructing a s.c. graph with 13 vertices. Two positive integers $p_2 = 4$ and $p_3 = 8$ are chosen. We note that the positive integers p_2 and p_3 are such that $p_2 + p_3 = 12$, $p_2 \leq p_3$, $p_2 = 2^2$ and $p_3 = 2^3$. Let $\sigma = (v_0)(v_{21}v_{22}v_{23}v_{24})(v_{31}v_{32}v_{33}v_{34}v_{35}v_{36}v_{37}v_{38})$ be a permutation on 13 labels v_0 and v_{ij} where $i = 2, 3$, $1 \leq j \leq p_i$. For the label v_0 the labels v_{21} and v_{31} constitute its range. For the label v_{21} the labels $v_{22}, v_{23}, v_{31}, v_{32}, v_{33}$ and v_{34} constitute its range. For the label v_{31} the labels v_{32}, v_{33}, v_{34} and v_{35} constitute its range. We construct a graph G with 13 vertices by identifying the vertices of G with the labels v_0 and v_{ij} where $i = 2, 3$, $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_0, v_{i1}]$ where v_{i1} is in the range of v_0 we decide arbitrarily whether $[v_0, v_{i1}] \in E(G)$ or $[v_0, v_{i1}] \notin E(G)$. Let the vertex v_0 be adjacent to v_{31} of its range and non-adjacent to v_{21} of its range as shown in Figure 3.2(a).

(b) After completing (a) whether the pair $[\sigma^k(v_0), \sigma^k(v_{i1})] \in E(G)$ or $[\sigma^k(v_0), \sigma^k(v_{i1})] \notin E(G)$ for all $1 \leq k \leq p_i - 1$ is determined as follows for each pair $[v_0, v_{i1}]$ such that v_{i1} is in the

range of v_0 . The pair $[\sigma^k(v_0), \sigma^k(v_{i1})] \in E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is even and $[\sigma^k(v_0), \sigma^k(v_{i1})] \notin E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is odd.

(c) For each pair $[v_{i1}, v_{jr}]$ where $i = 2, 3$ and v_{jr} is a vertex in the range of v_{i1} , we decide arbitrarily whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$. Let the vertex v_{21} be adjacent to the vertices v_{32}, v_{33} and v_{34} and non-adjacent to the vertices v_{22}, v_{23} and v_{31} of its range as shown in Figure 3.2(a). Let the vertex v_{31} be adjacent to the vertices v_{32}, v_{33} and v_{35} and non-adjacent to the vertex v_{34} of its range as shown in Figure 3.2(a).

(d) After completing (c) whether $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ or $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ where $i = 2, 3$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and $[\sigma^k(v_{i1}), \sigma^k(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The labelled graph G constructed is shown in Figure 3.2(b). We note that σ is a c.p. of the s.c. graph G .

3.3 On the existence of self-complementary chordal graphs

The following result was obtained independently by Ringel [178] and Sachs [190] for the existence of s.c. graphs.

Theorem 3.3 (Ringel [178], Sachs [190]) : *A self-complementary graph with p vertices exists if and only if $p=4n$ or $p=4n+1$ where n is a positive integer.*

The following result gives a necessary condition for the existence of s.c. chordal graphs.

Theorem 3.4 : *A s.c. chordal graph with p vertices exists only if $p=4n$ or $p=4n+1$ where n is a positive integer.*

Proof: Follows from Theorem 3.3. \square

Existence of s.c. chordal graphs with $4n$ vertices for all positive integers n is assured by the following Theorem.

Theorem 3.5 : *There exists s.c. chordal graphs with $4n$ vertices for all positive integer n .*

Proof: Let n be a positive integer. Let G be a graph with $4n$ vertices $0, 1, \dots, 4n - 1$ whose adjacencies are defined by the following conditions (a), (b) and (c).

- (a) Any two even labelled vertices are adjacent.
- (b) Any two odd labelled vertices are non-adjacent.
- (c) The vertex $2i + 1$ where $0 \leq i \leq 2n - 1$ is adjacent to the vertices $(2i + 2)_{4n}, (2i + 4)_{4n}, \dots, (2i + 2n)_{4n}$ and non-adjacent to the vertices $(2i + 2n + 2)_{4n}, (2i + 2n + 4)_{4n}, \dots, (2i + 4n)_{4n}$ where for a non-negative integer k the non-negative integer $(k)_{4n}$ is such that $k \equiv (k)_{4n} \pmod{4n}$.

The graphs G when $n=1$, $n=2$ and $n=3$ are shown in Figure 3.3(a), Figure 3.3(b) and Figure 3.3(c) respectively.

The graph G is a s.c. graph since $(0 \ 1 \ 2 \ \dots \ 4n - 1)$ is a c.p. of G . Let $V_1 = \{0, 2, 4, \dots, 4n - 2\}$ and $V_2 = \{1, 3, \dots, 4n - 1\}$. We note that $\langle V_1 \rangle$ is a complete subgraph of G and V_2 is a stable set of G . Also the sets V_1 and V_2 partition $V(G)$. Hence G is a split graph. So by Theorem 2.19 G is a s.c. chordal graph. \square

The existence of s.c. chordal graphs with $4n+1$ vertices for all positive integer n is assured by the following Theorem.

Theorem 3.6 : *There exist s.c. chordal graphs with $4n+1$ vertices for all positive integers n .*

Proof: Let n be a positive integer. Let G be a graph with $4n+1$ vertices $0, 1, 2, \dots, 4n$ whose adjacencies are defined by the following conditions (a), (b), (c) and (d).

- (a) Any two even labelled vertices are adjacent.
- (b) Any two odd labelled vertices are non-adjacent.
- (c) The vertex $4n$ is non-adjacent to the vertices $1, 3, \dots, 4n - 1$.
- (d) The vertex $2i + 1$ where $0 \leq i \leq 2n - 1$ is adjacent to the vertices $(2i + 2)_{4n}, (2i + 4)_{4n}, \dots, (2i + 2n)_{4n}$ and non-adjacent to the vertices $(2i + 2n + 2)_{4n}, (2i + 2n + 4)_{4n}, \dots, (2i + 4n)_{4n}$ where for a non-negative integer k the non-negative integer $(k)_{4n}$ is such that $k \equiv (k)_{4n} \pmod{4n}$.

The graphs G when $n=1$, $n=2$ and $n=3$ are shown in Figure 3.4(a), Figure 3.4(b) and Figure 3.4(c).

The graph G is a s.c. graph since $(0 \ 1 \ 2 \ \dots \ 4n-1)(4n)$ is a c.p. of G . Let $V_1 = \{0, 2, 4, \dots, 4n\}$ and $V_2 = \{1, 3, \dots, 4n-1\}$. We note that $\langle V_1 \rangle$ is a complete subgraph of G and V_2 is a stable set of G . Also V_1 and V_2 partition $V(G)$. Hence G is a split graph. Therefore by Theorem 2.19 G is a chordal graph. \square

The following result gives a necessary and sufficient condition for the existence of a s.c. chordal graph.

Theorem 3.7 : *A s.c. chordal graph with p vertices exists if and only if $p=4n$ or $p=4n+1$ where n is a positive integer.*

Proof: Follows from Theorem 3.4, Theorem 3.5 and Theorem 3.6. \square

3.4 Algorithms for constructing self-complementary chordal graphs

The following algorithm constructs all s.c. chordal graphs with $4n$ vertices.

Algorithm 3.3 : *An algorithm to construct s.c. chordal graphs with $4n$ vertices where n is a positive integer*

Choose positive integers p_1, p_2, \dots, p_s such that $\sum_{i=1}^s p_i = 4n$, $p_i \leq p_j$ for all $1 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $1 \leq i \leq s$. Let $\sigma^ = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $1 \leq i \leq s$ be a permutation on $4n$ labels $v_{11}, v_{12}, \dots, v_{1p_1}, v_{21}, v_{22}, \dots, v_{2p_2}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. For the label v_{i1} where $1 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, v_{(i+2)1}, v_{(i+2)2}, \dots, v_{(i+2)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_i}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. Construct a graph G with $4n$ vertices by identifying the vertices of G with the labels v_{ij} where $1 \leq i \leq s$ and $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.*

(a) *For each pair $[v_{i1}, v_{jr}]$ where $1 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} the pair $[v_{i1}, v_{jr}] \notin E(G)$ when r is odd and it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$ when r is even.*

(b) *After completing (a) whether the pairs $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ or $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$*

$E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ such that $1 \leq i \leq s$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and the pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The following Theorem is a characterisation of a s.c. graph G with $4n$ vertices to be a chordal graph in terms of its induced subgraph $\langle \text{Even}(\sigma^*) \rangle$ where σ^* is a star c.p. of G .

Theorem 3.8 : *Let G be a s.c. graph with $p=4n$ and a star c.p. σ^* . Then G is a chordal graph if and only if $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G .*

Proof: Let G be a chordal graph. By Theorem2.20 $\omega(G) = 2n$. Hence by Theorem2.14 $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G .

Let $\langle \text{Even}(\sigma^*) \rangle$ be a complete subgraph of G . We note that $\omega(G) \geq 2n$ since $|\text{Even}(\sigma^*)| = 2n$. Hence by Theorem2.11 $\omega(G) = 2n$. By Theorem2.20 it follows that G is a chordal graph. \square

The following Theorem states that a graph G with $4n$ vertices is a s.c. chordal graph if and only if it can be constructed by Algorithm3.3.

Theorem 3.9 : *(i) The graphs constructed by Algorithm3.3 are s.c. chordal graphs with $4n$ vertices where n is a positive integer.*

(ii) Every s.c. chordal graph with $4n$ vertices for all positive integers n can be constructed by Algorithm3.3.

Proof: (i) Let n be a positive integer. Let p_1, p_2, \dots, p_s be positive integers such that $\sum_{i=1}^s p_i = 4n$, $p_i \leq p_j$ for all $1 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $1 \leq i \leq s$. Let $\sigma^* = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ where $1 \leq i \leq s$ be a permutation on $4n$ labels $v_{11}, v_{12}, \dots, v_{1p_1}, v_{21}, v_{22}, \dots, v_{2p_2}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. Let G be a graph constructed by Algorithm3.3 from the permutation σ^* . We note that G can also be constructed by Algorithm3.1 from the permutation σ^* . Hence by Theorem3.1(i) G is a s.c. graph. By the definition of the graph G it follows that σ^* is a star c.p. of G . For the label v_{i1} where

$1 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_i}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. We note that for any two vertices v_{i1} and v_{jm} belonging to $Even(\sigma^*)$ where $1 \leq i < j \leq s$ the pair $[v_{i1}, v_{jm}] = [\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jm})]$ for some odd integers k and r such that $1 \leq k \leq p_j - 1$ and v_{jr} is in the range of v_{i1} . By the definition of the graph G , $[v_{i1}, v_{jr}] \notin E(G)$ for each pair $[v_{i1}, v_{jr}]$ where $1 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} such that r is odd. Also by the definition of G the pair $[v_{i1}, v_{jr}] \notin E(G)$ implies $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ when k is odd such that $1 \leq k \leq p_j - 1$ for each pair $[v_{i1}, v_{jr}]$ where $1 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} such that r is odd. So $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G . By Theorem3.8 it follows that G is a chordal graph.

(ii) Let G be a s.c. chordal graph with $p=4n$ where n is a positive integer. Let $\sigma^* = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $1 \leq i \leq s$ be a star c.p. of G . By Theorem3.8 $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G . So $Odd(\sigma^*)$ is a stable set of G since $Odd(\sigma^*) = \sigma^*(Even(\sigma^*))$. Then by Lemma2.2 it follows that G can be constructed by Algorithm3.3 from the permutation σ^* . \square

We illustrate Algorithm3.3 by constructing a s.c. chordal graph with 12 vertices. Two positive integers $p_1 = 4$ and $p_2 = 8$ are chosen. We note that the positive integers p_1 and p_2 are such that $p_1 + p_2 = 12$, $p_1 \leq p_2$, $p_1 = 2^2$ and $p_2 = 2^3$. Let $\sigma^* = (v_{11} v_{12} v_{13} v_{14})(v_{21} v_{22} v_{23} v_{24} v_{25} v_{26} v_{27} v_{28})$ be a permutation on 12 labels v_{ij} where $i = 1, 2$ and $1 \leq j \leq p_i$. For the label v_{11} the labels $v_{12}, v_{13}, v_{21}, v_{22}, v_{23}$ and v_{24} constitute its range. For the label v_{21} the labels v_{22}, v_{23}, v_{24} and v_{25} constitute its range. We construct a graph G with 12 vertices by identifying the vertices of G with the labels v_{ij} where $i = 1, 2$ and $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_{i1}, v_{jr}]$ where $i = 1, 2$ and v_{jr} is a vertex in the range of v_{i1} the pair $[v_{i1}, v_{jr}] \notin E(G)$ when r is odd and it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$ when r is even. Let the vertex v_{11} be adjacent to the vertices v_{12} and v_{22} and non-adjacent to the vertex v_{24} of its range as shown Figure 3.5(a). Let the vertex v_{21} be adjacent to the vertex v_{24} and non-adjacent to the vertex v_{22} of its range as shown in Figure3.5(a).

(b) After completing (a) whether $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ or $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$

for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ where $i = 1, 2$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The labelled graph G constructed is shown in Figure 3.5(b). We note that σ^* is a star c.p. of the s.c. chordal graph G .

Remark: Algorithm3.3 may construct the same graph (with different labellings) repetitively. For example consider the graphs shown in Figure 3.6(b) and Figure 3.7(b) which are constructed by Algorithm3.3 with the adjacency schemes shown in Figure 3.6(a) and Figure 3.7(a) respectively. We note that these two graphs are isomorphic. So after constructing all s.c. chordal graphs with $4n$ vertices from Algorithm3.3 to compile a catalogue of s.c. chordal graphs with $4n$ vertices every pair of the constructed graphs should be tested for isomorphism.

The following algorithm constructs all s.c. chordal graphs with $4n+1$ vertices.

Algorithm 3.4 : An algorithm to construct s.c. chordal graphs with $4n+1$ vertices, n a positive integer

Choose positive integers p_2, p_3, \dots, p_s such that $\sum_{i=2}^s p_i = 4n$, $p_i \leq p_j$ for all $2 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $2 \leq i \leq s$. Let $\sigma^ = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_1^* = (v_0)$ and $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $2 \leq i \leq s$ be a permutation on $4n+1$ labels $v_0, v_{21}, v_{22}, \dots, v_{2p_2}, v_{31}, v_{32}, \dots, v_{3p_3}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. For the label v_0 define $v_{21}, v_{31}, \dots, v_{s1}$ as its range. For the label v_{i1} where $2 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, v_{(i+2)1}, v_{(i+2)2}, \dots, v_{(i+2)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_i}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. Construct a graph G with $4n+1$ vertices by identifying the vertices of G with the labels v_0 and v_{ij} where $2 \leq i \leq s$, $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.*

(a) For each pair $[v_0, v_{i1}]$ where v_{i1} is in the range of v_0 the pair $[v_0, v_{i1}] \notin E(G)$.

(b) After completing (a) whether the pairs $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \in E(G)$ or $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \notin E(G)$ for all $1 \leq k \leq p_i - 1$ is determined as follows for each pair $[v_0, v_{i1}]$ such that v_{i1} is in the range of v_0 . The pair $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \notin E(G)$ when k is even and $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \in E(G)$ when k is odd.

(c) For each pair $[v_{i1}, v_{jr}]$ where $2 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} the pair $[v_{i1}, v_{jr}] \notin E(G)$ when r is odd and it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$ when r is even.

(d) After completing (c) whether the pairs $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ or $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ such that $2 \leq i \leq s$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and the pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

Let G be a s.c. graph with a c.p. σ . The following Theorem establishes a relation between the degree of a vertex and the degree of its image under σ .

Theorem 3.10 : Let G be a s.c. graph with p vertices. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$ where $\sigma_i = (v_{i1} v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$ be a c.p. of G . Then $\deg_G(v_{ij}) + \deg_G(\sigma(v_{ij})) = p - 1$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$.

Proof: We note that $\deg_G(v_{ij}) = \deg_{\tilde{G}}(\sigma(v_{ij}))$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$ since σ is a vertex isomorphism of G onto \tilde{G} . Also $\deg_{\tilde{G}}(\sigma(v_{ij})) = p - 1 - \deg_G(\sigma(v_{ij}))$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. Hence $\deg_G(v_{ij}) + \deg_G(\sigma(v_{ij})) = p - 1$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. \square

Corollary 3.1 : Let G be a s.c. chordal graph with $p=4n+1$. Let σ be a c.p. of G with v_0 as its fixed vertex. Then $\deg_G(v_0) = 2n$.

Proof: We note that $\sigma(v_0) = v_0$ since v_0 is the fixed vertex of σ . Hence from Theorem 3.10 $\deg_G(v_0) = 2n$. \square

Let G be a s.c. chordal graph with $4n+1$ vertices. Let σ be a c.p. of G with v_0 as its fixed vertex. The following Theorem gives the structure of the induced subgraph of G induced by the vertices adjacent to v_0 .

Theorem 3.11 : Let G be a s.c. chordal graph with $p=4n+1$ and a c.p. σ . Let v_0 be the fixed vertex of σ . Then $\langle Nhd_G(v_0) \rangle \cong K_{2n}$.

Proof: By Theorem 2.20 $\omega(G) = 2n + 1$. Let C be a maximum clique of G . Let V_C be the vertex set of C . We prove that $v_0 \in V_C$. Let $v_0 \notin V_C$. We note that C is a maximum clique of $G - v_0$. Then $\omega(G - v_0) = 2n + 1$. By Corollary 2.1 $G - v_0$ is a s.c. graph. So by Theorem 2.11 $\omega(G - v_0) \leq 2n$, a contradiction. Hence $v_0 \in V_C$. By Corollary 3.1 $\deg_G(v_0) = 2n$. Then it follows that $\langle Nhd_G(v_0) \rangle \cong K_{2n}$. \square

Let G be a s.c. chordal graph with $4n+1$ vertices. Let σ^* be a star c.p. of G with v_0 as its fixed vertex. The following Theorem states that the vertices adjacent to v_0 are precisely the vertices in $Even(\sigma^*)$.

Theorem 3.12 : *Let G be a s.c. chordal graph with $p=4n+1$ and a star c.p. σ^* . Let v_0 be the fixed vertex of σ^* . Then $Nhd_G(v_0) = Even(\sigma^*)$.*

Proof: Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$. By Corollary 3.1 $|Nhd_G(v_0)| = 2n$. So to show that $Nhd_G(v_0) = Even(\sigma^*)$, we prove that $Nhd_G(v_0) \cap (Odd(\sigma^*) - \{v_0\}) = \emptyset$ for both the sets $Nhd(v_0)$ and $Even(\sigma^*)$ have $2n$ elements. Let a vertex $v_{ij} \in Nhd_G(v_0) \cap (Odd(\sigma^*) - \{v_0\})$. By Corollary 2.2 $G - v_0$ is a s.c. graph with a star c.p. σ^*/σ_1^* . So by Theorem 2.11 and Theorem 3.11 $\omega(G - v_0) = 2n$. By Theorem 2.14 $\langle Even(\sigma^*/\sigma_1^*) \rangle$ is a complete subgraph of $G - v_0$ and hence a complete subgraph of G . It follows that $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G since $Even(\sigma^*) = Even(\sigma^*/\sigma_1^*)$. Since the induced subgraph $\langle Even(\sigma^*) \rangle$ of G is a complete subgraph of G , it implies that $Odd(\sigma^*) - \{v_0\}$ is a stable set of G since $Odd(\sigma^*) - \{v_0\} = \sigma^*(Even(\sigma^*))$. The vertex v_{ij} belongs to $Odd(\sigma^*) - \{v_0\}$ implies that $\sigma^{*2}(v_{ij})$ is a vertex distinct from v_{ij} such that $\sigma^{*2}(v_{ij}) \in Odd(\sigma^*) - \{v_0\}$ since by Theorem 2.1 the length of every permutation cycle of σ^* is atleast 4 except the cycle (v_0) . We note that $[v_{ij}, v_0] \in E(G)$ since $v_{ij} \in Nhd_G(v_0)$. Also by Lemma 2.2 $[v_{ij}, v_0] \in E(G)$ implies $[\sigma^{*2}(v_{ij}), \sigma^{*2}(v_0)] = [\sigma^{*2}(v_{ij}), v_0] \in E(G)$. Then $\sigma^{*2}(v_{ij}) \in Nhd_G(v_0)$. The pair $[v_{ij}, \sigma^{*2}(v_{ij})] \notin E(G)$ since $v_{ij} \in Odd(\sigma^*) - \{v_0\}$, $\sigma^{*2}(v_{ij}) \in Odd(\sigma^*) - \{v_0\}$ and $Odd(\sigma^*) - \{v_0\}$ is a stable set of G . Hence $Nhd_G(v_0)$ is not a complete subgraph of G , a contradiction to Theorem 3.11. Therefore $Nhd_G(v_0) \cap (Odd(\sigma^*) - \{v_0\}) = \emptyset$. \square

The following Theorem is a characterisation of a s.c. graph G with $4n+1$ vertices to be a chordal graph in terms of its induced subgraph $\langle Even(\sigma^*) \cup \{v_0\} \rangle$ where σ^* is a star c.p. of G with v_0 as its fixed vertex.

CENTRAL LIBRARY
I. I. T., KANPUR

NO. A 131082

Theorem 3.13 : *Let G be a s.c. graph with $4n+1$ vertices. Let σ^* be a star c.p. of G with v_0 as its fixed vertex. Then G is chordal if and only if $\langle \text{Even}(\sigma^*) \cup \{v_0\} \rangle$ is a complete subgraph of G .*

Proof: Let G be a chordal graph. By Theorem3.11 and Theorem3.12 $\langle \text{Even}(\sigma^*) \cup \{v_0\} \rangle$ is a complete subgraph of G .

Let $\langle \text{Even}(\sigma^*) \cup \{v_0\} \rangle$ be a complete subgraph of G . Then $\omega(G) \geq 2n + 1$ since $|\text{Even}(\sigma^*)| = 2n$. So by Theorem2.11 it follows that $\omega(G) = 2n + 1$. Hence by Theorem2.20 G is a chordal graph. \square

The following Theorem states that a graph G with $4n+1$ vertices is a s.c. chordal graph if and only if it can be constructed by Algorithm3.4.

Theorem 3.14 : *(i) The graphs constructed by Algorithm3.4 are s.c. chordal graphs with $4n+1$ vertices where n is a positive integer.*

(ii) Every s.c. chordal graph with $4n+1$ vertices for all positive integers n can be constructed by Algorithm3.4.

Proof: (i) Let n be a positive integer. Let p_2, p_3, \dots, p_s be positive integers such that $\sum_{i=2}^s p_i = 4n$, $p_i \leq p_j$ for all $2 \leq i < j \leq s$ and $p_i = 2^{k_i}$ for some $k_i \geq 2$ where $2 \leq i \leq s$. Let $\sigma^* = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_1^* = (v_0)$ and $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ where $2 \leq i \leq s$ be a permutation on $4n+1$ labels $v_0, v_{21}, v_{22}, \dots, v_{2p_2}, v_{31}, v_{32}, \dots, v_{3p_3}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$. Let G be a graph constructed by Algorithm3.4 from the permutation σ^* . We note that G can also be constructed by Algorithm3.2 from the permutation σ^* . Hence by Theorem3.2(i) G is a s.c. graph. By the definition of the graph G it follows that σ^* is a star c.p. of G . For the label v_{i1} where $2 \leq i \leq s-1$ define $v_{i2}, v_{i3}, \dots, v_{i(\frac{p_i}{2}+1)}, v_{(i+1)1}, v_{(i+1)2}, \dots, v_{(i+1)p_i}, \dots, v_{s1}, v_{s2}, \dots, v_{sp_s}$ as the range of v_{i1} . For the label v_{s1} define $v_{s2}, v_{s3}, \dots, v_{s(\frac{p_s}{2}+1)}$ as its range. We note that for any two vertices v_{i1} and v_{jr} belonging to $\text{Even}(\sigma^*)$ where $2 \leq i < j \leq s$ the pair $[v_{i1}, v_{jr}] = [\sigma^{*k}(v_{i1}), \sigma^{*r}(v_{jr})]$ for some odd integers k and r such that $1 \leq k \leq p_j - 1$ and v_{jr} is in the range of v_{i1} . By the definition of the graph G , $[v_{i1}, v_{jr}] \notin E(G)$ for each pair $[v_{i1}, v_{jr}]$ where $2 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} such that r is odd. Also by the definition of G the pair $[v_{i1}, v_{jr}] \notin E(G)$ implies $[\sigma^{*k}(v_{i1}), \sigma^{*r}(v_{jr})] \in E(G)$ when k is

odd such that $1 \leq k \leq p_j - 1$ for each pair $[v_{i1}, v_{jr}]$ where $2 \leq i \leq s$ and v_{jr} is a vertex in the range of v_{i1} such that r is odd. So $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . By the definition of G , $[v_0, v_{i1}] \notin E(G)$ for all $2 \leq i \leq s$. Also for all $v_{ij} \in \text{Even}(\sigma^*)$ where $2 \leq i \leq s$ the pair $[v_0, v_{ij}] = [\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})]$ where k is some odd integer such that $1 \leq k \leq p_i - 1$. So $[v_0, v_{ij}] \in E(G)$ for all $v_{ij} \in \text{Even}(\sigma^*)$ since $[v_0, v_{i1}] \notin E(G)$ for all $2 \leq i \leq s$. Therefore $\langle \{v_0\} \cup \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . By Theorem3.13 it follows that G is a chordal graph.

(ii) Let G be a s.c. chordal graph with $p=4n+1$ where n is a positive integer. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_1^* = (v_0)$ and $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for all $2 \leq i \leq s$ be a star c.p. of G . By Theorem3.13 $\langle \{v_0\} \cup \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . So $\text{Odd}(\sigma^*) \cup \{v_0\}$ is a stable set of G since $\text{Odd}(\sigma^*) \cup \{v_0\} = \sigma^*(\text{Even}(\sigma^*) \cup \{v_0\})$. Then by Lemma2.2 it follows that G can be constructed by Algorithm3.4 from the permutation σ^* . \square

We illustrate Algorithm3.4 by constructing a s.c. chordal graph with 13 vertices. Two positive integers $p_2 = 4$ and $p_3 = 8$ are chosen. We note that $p_2 + p_3 = 12$, $p_2 \leq p_3$, $p_2 = 2^2$ and $p_3 = 2^3$. Let $\sigma^* = (v_0)(v_{21} v_{22} v_{23} v_{24})(v_{31} v_{32} v_{33} v_{34} v_{35} v_{36} v_{37} v_{38})$ be a permutation on 13 labels v_0 and v_{ij} where $i = 2, 3$, $1 \leq j \leq p_i$. For the label v_0 the labels v_{21} and v_{31} constitute its range. For the label v_{21} the labels $v_{22}, v_{23}, v_{31}, v_{32}, v_{33}$ and v_{34} constitute its range. For the label v_{31} the labels v_{32}, v_{33}, v_{34} and v_{35} constitute its range. We construct a graph G with 13 vertices by identifying the vertices of G with the labels v_0 and v_{ij} where $i = 2, 3$, $1 \leq j \leq p_i$ and the adjacencies of the vertices of G are determined as follows.

(a) For each pair $[v_0, v_{i1}]$ where v_{i1} is in the range of v_0 the pair $[v_0, v_{i1}] \notin E(G)$ as shown in Figure 3.8(a).

(b) After completing (a) whether the pair $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \in E(G)$ or $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \notin E(G)$ for all $1 \leq k \leq p_i - 1$ is determined as follows for each pair $[v_0, v_{i1}]$ such that v_{i1} is in the range of v_0 . The pair $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \in E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is even and $[\sigma^{*k}(v_0), \sigma^{*k}(v_{i1})] \notin E(G)$ if and only if $[v_0, v_{i1}] \in E(G)$ when k is odd.

(c) For each pair $[v_{i1}, v_{jr}]$ where $i = 2, 3$ and v_{jr} is a vertex in the range of v_{i1} the pair $[v_{i1}, v_{jr}] \notin E(G)$ when r is odd and it is arbitrarily decided whether $[v_{i1}, v_{jr}] \in E(G)$ or $[v_{i1}, v_{jr}] \notin E(G)$ when r is even. Let the vertex v_{21} be adjacent to the vertices v_{22} and v_{32} and non-adjacent to the vertex v_{34} of its range as shown in Figure 3.8(a). Let the vertex

v_{31} be adjacent to the vertex v_{34} and non-adjacent to the vertex v_{32} of its range as shown in Figure 3.8(a).

(d) After completing (c) whether $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ or $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ for all $1 \leq k \leq p_j - 1$ is determined as follows for each pair $[v_{i1}, v_{jr}]$ where $i = 2, 3$ and v_{jr} is in the range of v_{i1} . The pair $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \in E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is even and $[\sigma^{*k}(v_{i1}), \sigma^{*k}(v_{jr})] \notin E(G)$ if and only if $[v_{i1}, v_{jr}] \in E(G)$ when k is odd.

The labelled graph G constructed is shown in Figure 3.8(b). We note that σ^* is a star c.p. of the s.c. chordal graph G .

Remark: Algorithm3.4 may construct the same graph (with different labellings) many times. For example consider the graphs shown in Figure 3.9(b) and Figure 3.10(b) which are constructed by Algorithm3.4 with the adjacency schemes shown in Figure 3.9(a) and Figure 3.10(a) respectively. The two graphs constructed by Algorithm3.4 are isomorphic. To compile a catalogue of s.c. chordal graphs with $4n+1$ vertices, every pair of the constructed graphs should be tested whether the graphs are isomorphic or not after constructing all s.c. chordal graphs with $4n+1$ vertices from Algorithm3.4.

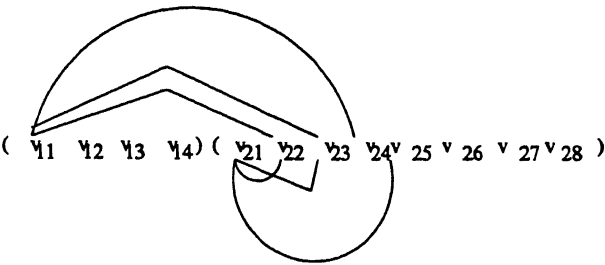


Figure 3.1a

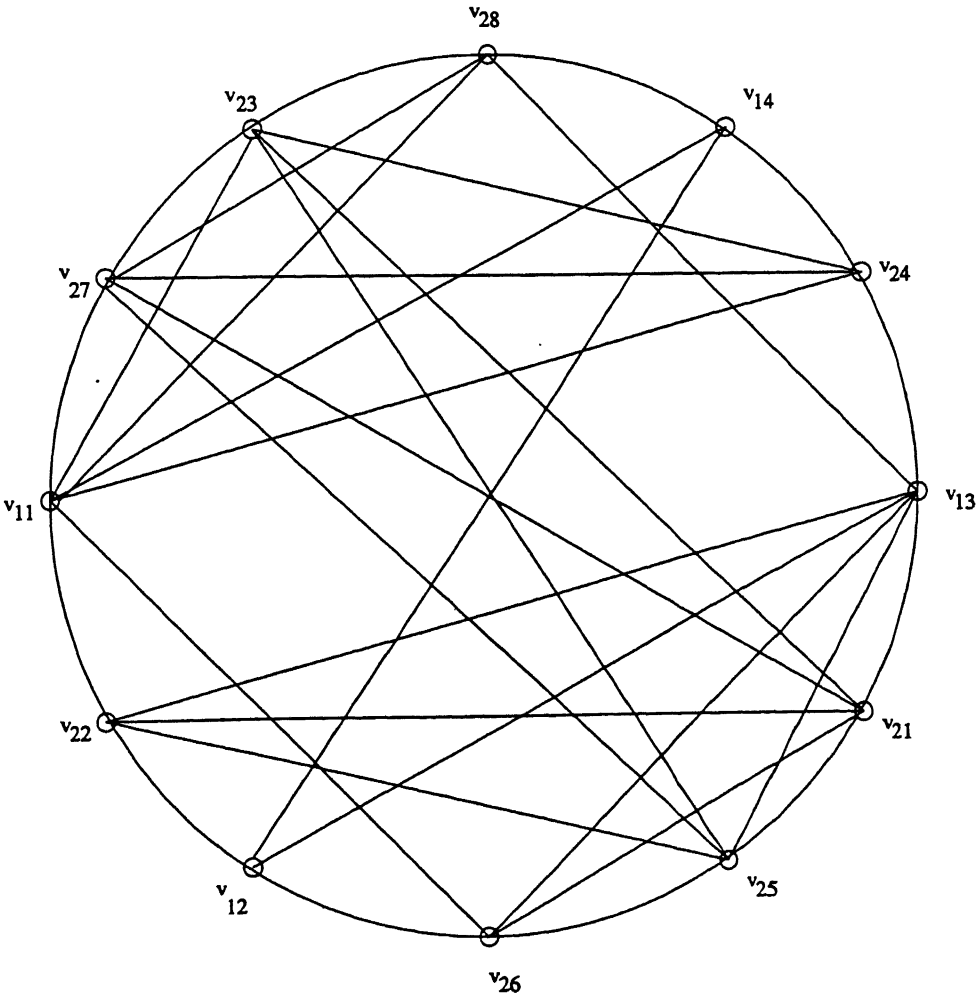


Figure 3.1b

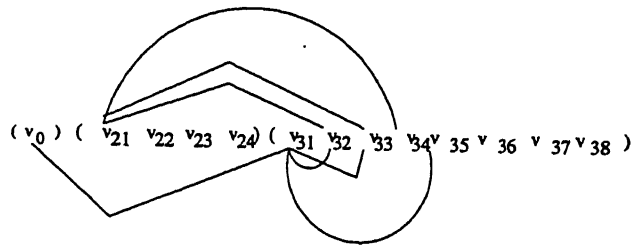


Figure 3.2a

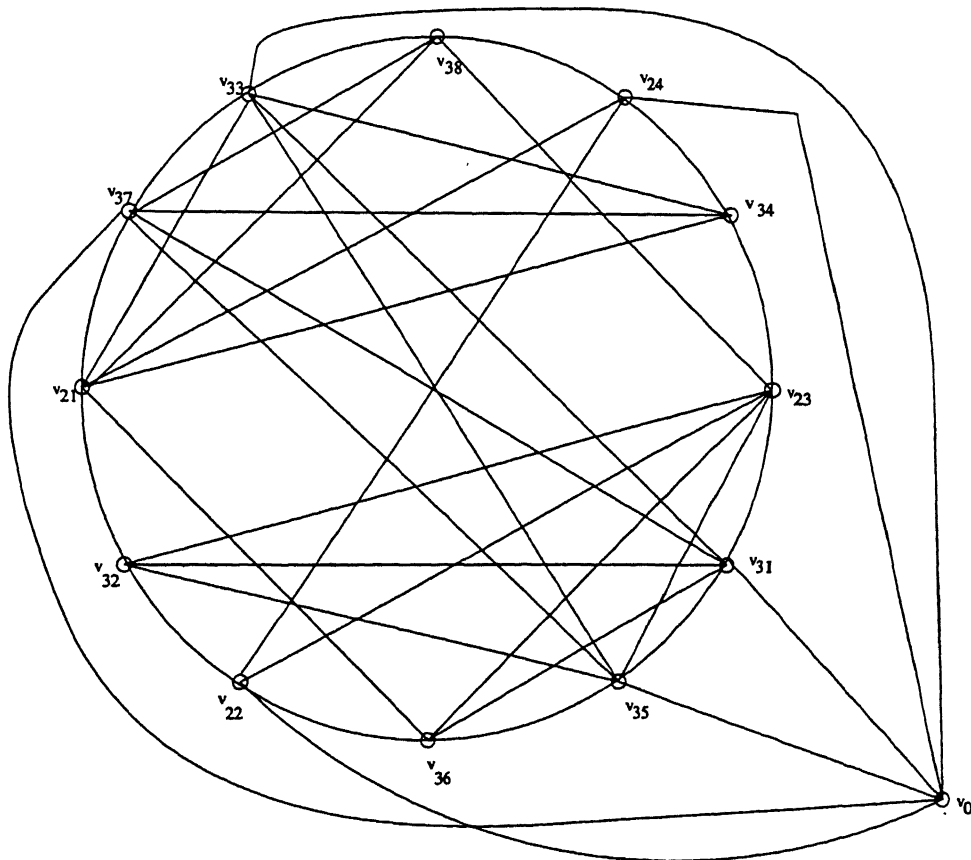


Figure 3.2b

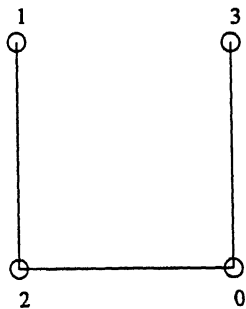


Figure 3.3a

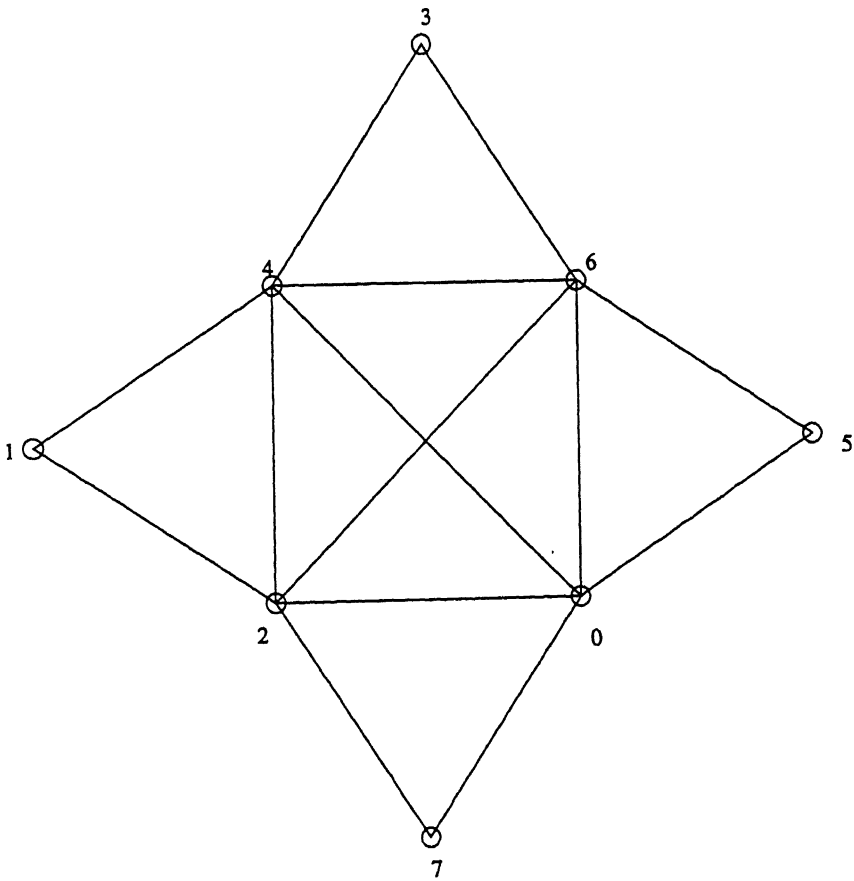


Figure 3.3b

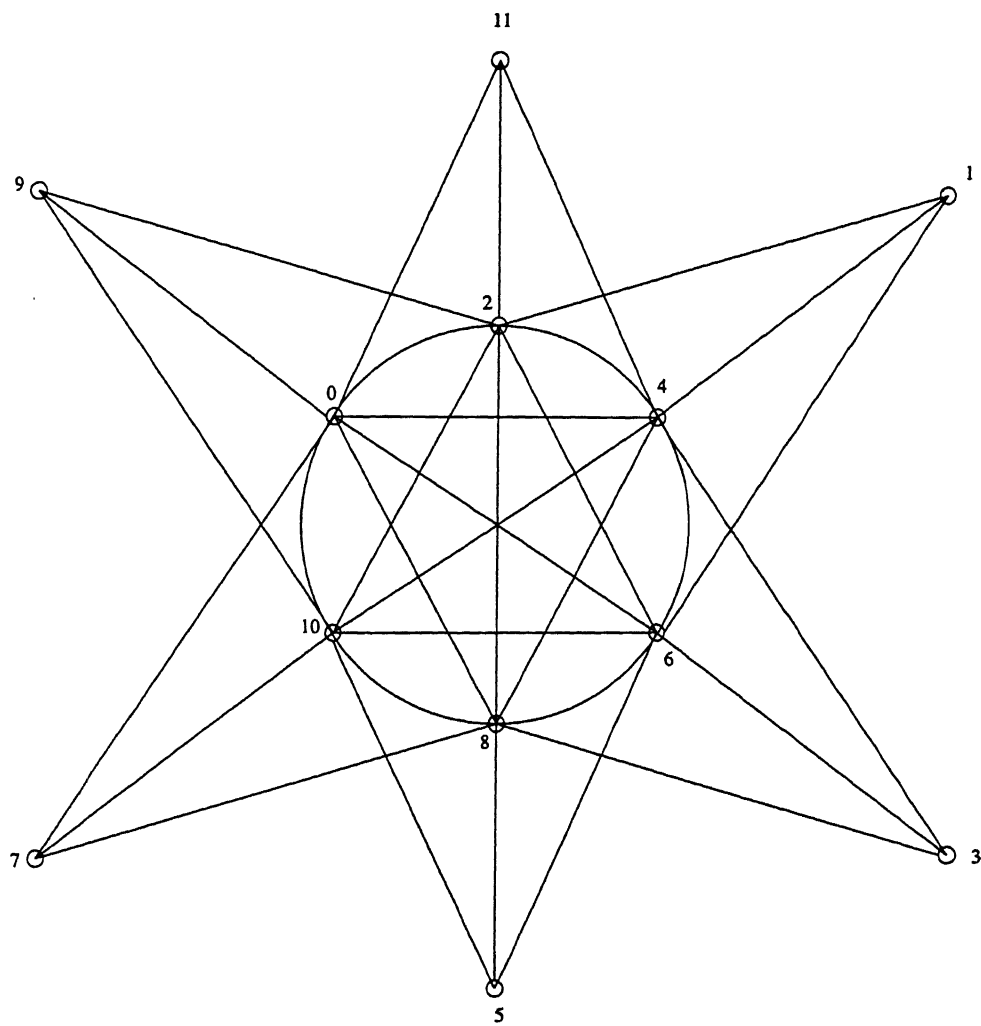


Figure 3.3c

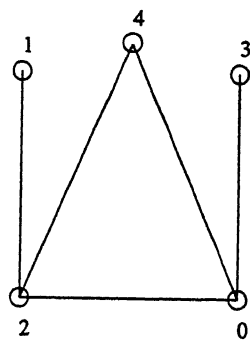


Figure 3.4a

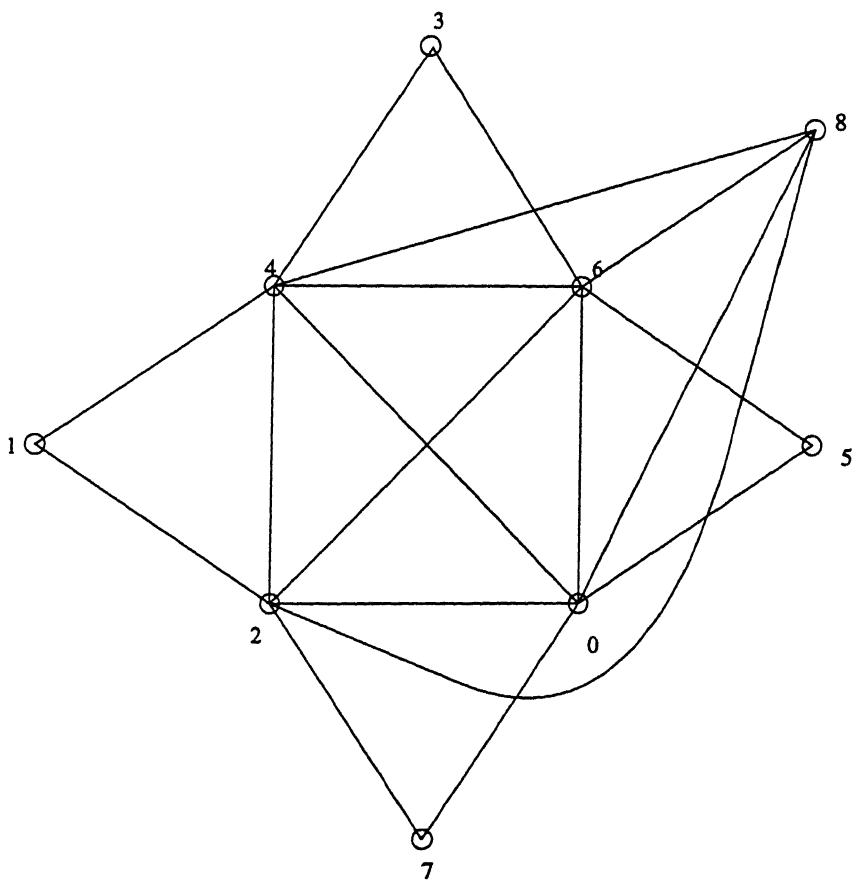


Figure 3.4b

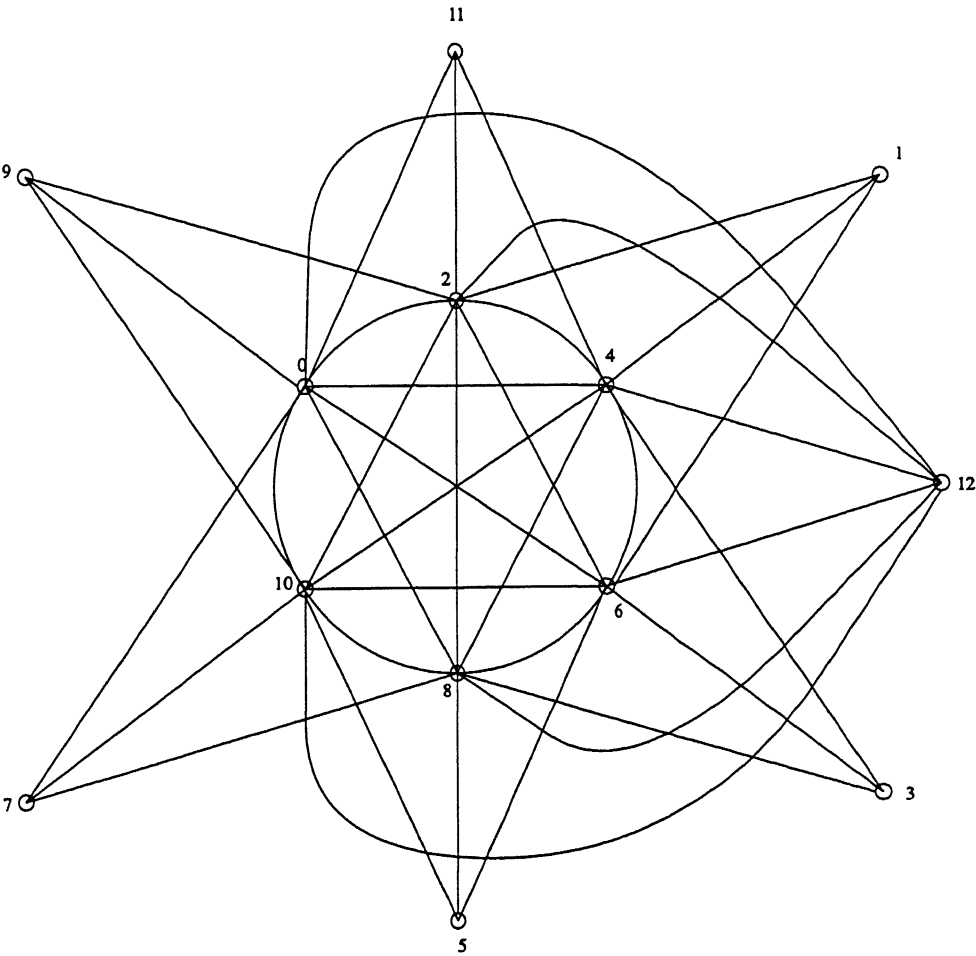


Figure 3.4c

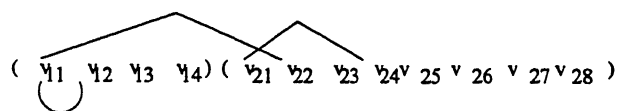


Figure 3.5a

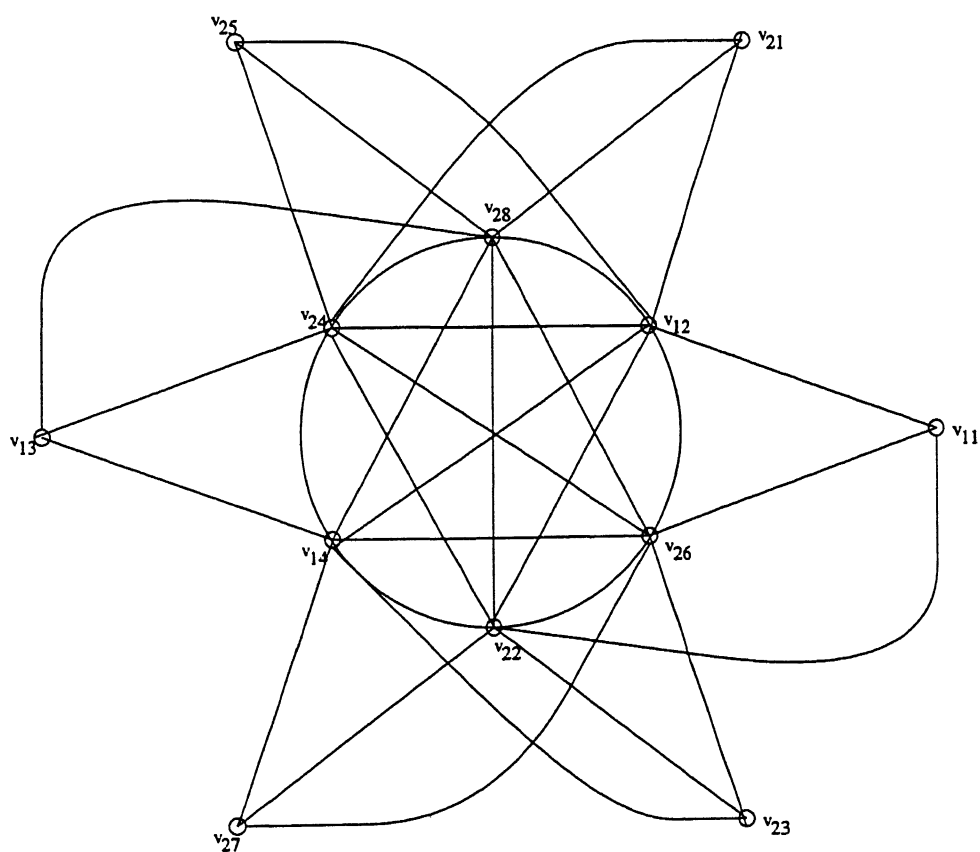


Figure 3.5b

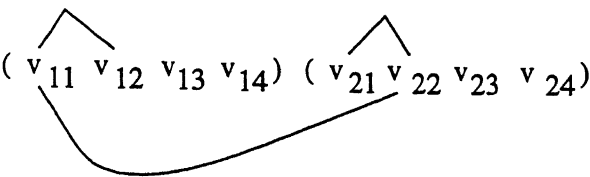


Figure 3.6a

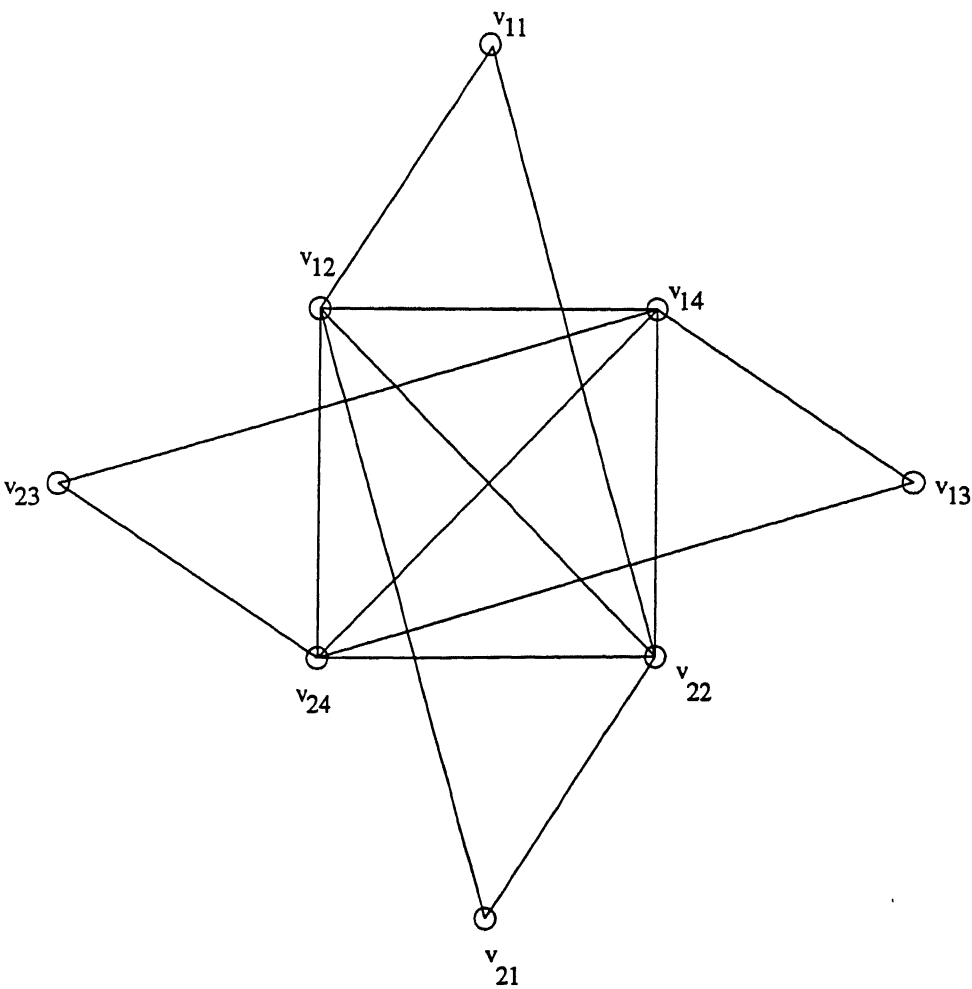


Figure 3.6b

($\overset{\frown}{v_{11} v_{12} v_{13} v_{14} v_{15} v_{16} v_{17} v_{18}}$)

Figure 3.7a

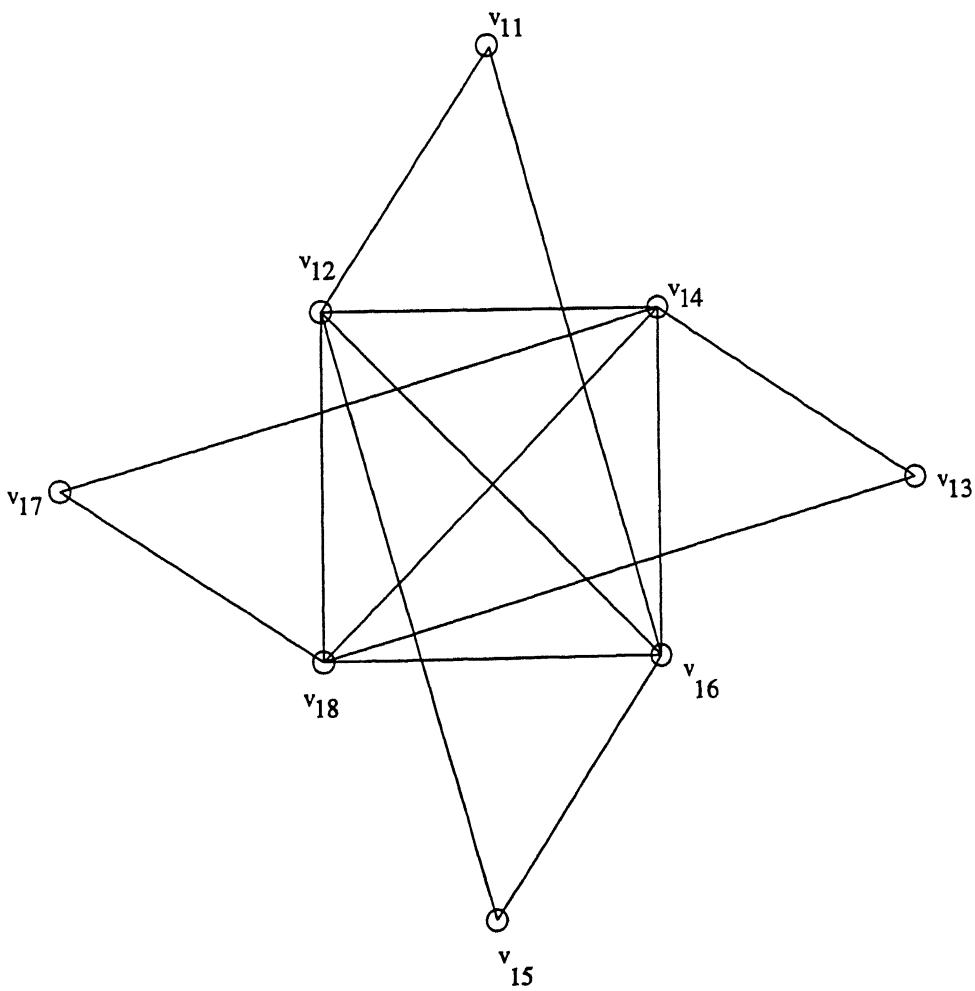


Figure 3.7b

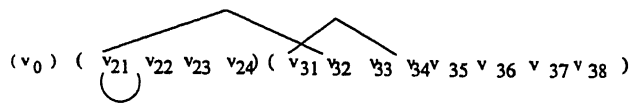


Figure 3.8a

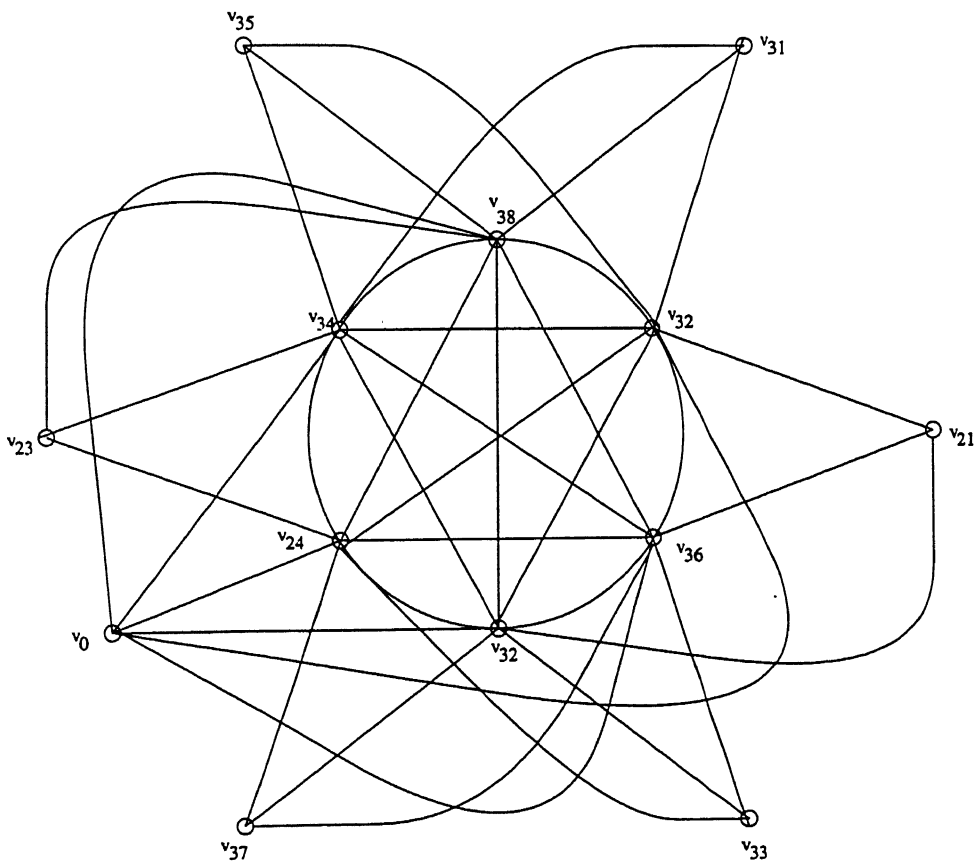


Figure 3.8b

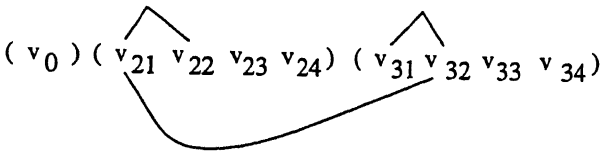


Figure 3.9a

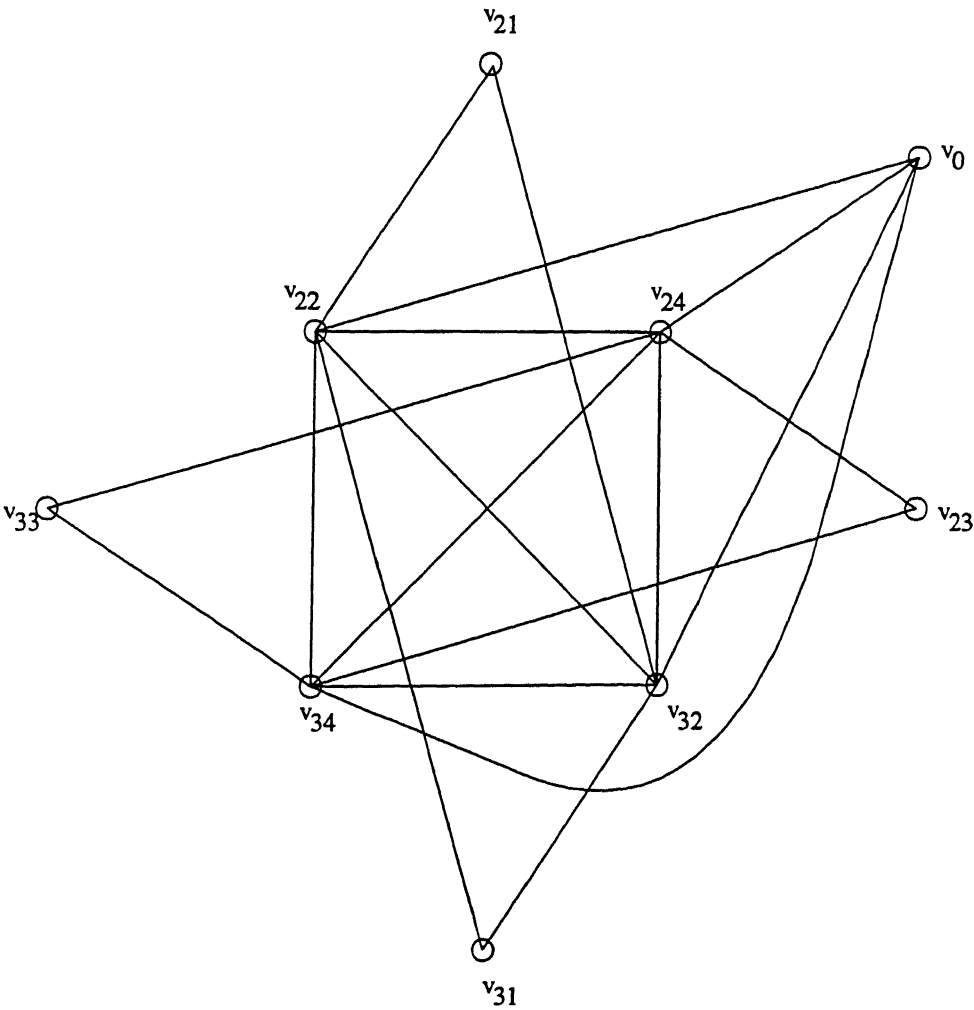


Figure 3.9b

$(v_0) (\overset{\wedge}{v_{21} v_{22} v_{23} v_{24} v_{25} v_{26} v_{27} v_{28}})$

Figure 3.10a

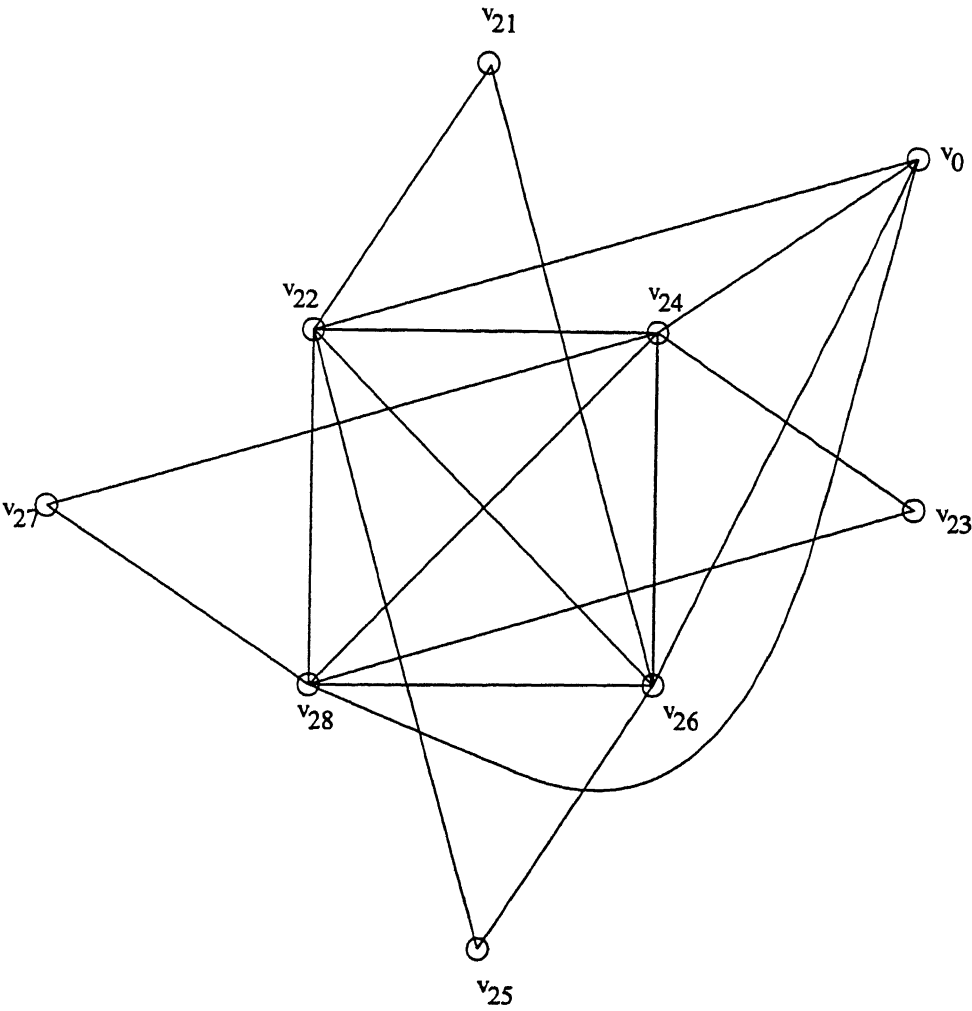


Figure 3.10b

Chapter 4

On the isomorphism and the catalogue compilation of self-complementary chordal graphs

4.1 Introduction

Colbourn et. al. [65] and [66] prove that the isomorphism of s.c. graphs and the isomorphism of regular s.c. graphs are polynomially equivalent to the isomorphism problem. In Section 4.2 of this chapter we prove that the isomorphism of s.c. chordal graphs, the isomorphism of s.c. chordal graphs with $4n$ vertices and the isomorphism of s.c. chordal graphs with $4n+1$ vertices are polynomially equivalent. Catalogue of s.c. graphs with 8 vertices had been compiled by Alter [5], Faradzhev [78], Morris [143] and Venkatachalam [225]. Faradzhev [78] and Morris [143] also compiled the catalogue of s.c. graphs with 9 vertices. Faradzhev [78] and Kropar et. al. [127] obtained the catalogue of s.c. graphs with 12 vertices. Kropar and Read in [127] observed that obtaining catalogue of s.c. graphs with more than 12 vertices was quite difficult. In the case of s.c. chordal graphs too the compilation of catalogue seems to be quite difficult. This difficulty is due to its close relationship with the isomorphism of s.c. chordal graphs as observed in the previous chapter. However we could compile the catalogue of s.c. chordal graphs with atmost 13 vertices using the existing catalogue of s.c.

graphs. In Section 4.3 of this chapter we give linear time algorithms to decide whether a given s.c. graph with $4n$ or $4n+1$ vertices is chordal or not. Using this algorithm, from the available catalogue of s.c. graphs with atmost 12 vertices we locate those graphs which are chordal and obtain the catalogue of s.c. chordal graphs with atmost 12 vertices. We also obtain the catalogue of s.c. chordal graphs with 13 vertices by giving a method for obtaining all non-isomorphic s.c. chordal graphs with $4n+1$ vertices from the set of all non-isomorphic s.c. chordal graphs with $4n$ vertices.

4.2 Algorithmic complexity of the isomorphism of self-complementary chordal graphs

Lemma 4.1 : *Let G' and G'' be such that $G' \cong G''$. Let ψ be a vertex isomorphism of G' onto G'' . Then $G' - v \cong G'' - \psi(v)$.*

Proof: Follows from the definition of ψ the vertex isomorphism of G' onto G'' . \square

For a s.c. graph G with $4n$ vertices and a star c.p. σ^* the following result gives an upper bound and a lower bound for the degrees of the vertices in $Odd(\sigma^*)$ and $Even(\sigma^*)$ respectively.

Theorem 4.1 : *Let G be a s.c. chordal graph with $p=4n$ and a star c.p. σ^* . Then*

(i) $deg_G(v_{ij}) \leq 2n - 1$ for all $v_{ij} \in Odd(\sigma^*)$

(ii) $deg_G(v_{ij}) \geq 2n$ for all $v_{ij} \in Even(\sigma^*)$.

Proof: (i) We note that $\sigma^*(Even(\sigma^*)) = Odd(\sigma^*)$. Also we note that $\omega(G) = 2n$ by Theorem 2.20. Let $deg_G(v_{ij}) \geq 2n$ for some vertex $v_{ij} \in Odd(\sigma^*)$. By Theorem 2.14 $\langle Even(\sigma^*) \rangle$ is a maximum clique of G . Hence $Odd(\sigma^*)$ is a stable set of G . So the vertex v_{ij} can be adjacent with only even labelled vertices of σ^* in G . We note that v_{ij} is adjacent with all even labelled vertices of σ^* since $deg_G(v_{ij}) \geq 2n$, $|Even(\sigma^*)| = 2n$ and v_{ij} is adjacent with only even labelled vertices of σ^* . Hence $\langle \{v_{ij}\} \cup Even(\sigma^*) \rangle$ is a complete subgraph of G , a contradiction. Therefore $deg_G(v_{ij}) \leq 2n - 1$ for all $v_{ij} \in Odd(\sigma^*)$.

(ii) Let $\deg_G(v_{ij}) \leq 2n - 1$ for some vertex $v_{ij} \in \text{Even}(\sigma^*)$. By Theorem3.10 $\deg_G(\sigma^*(v_{ij})) \geq 2n$, a contradiction to (i) of this Theorem since $\sigma^*(v_{ij}) \in \text{Odd}(\sigma^*)$. Therefore $\deg_G(v_{ij}) \geq 2n$ for all $v_{ij} \in \text{Even}(\sigma^*)$. \square

For a s.c. graph G with $4n+1$ vertices and a star c.p. σ^* the following result gives an upper bound and a lower bound for the degrees of the vertices in $\text{Odd}(\sigma^*) - \{v_0\}$ and $\text{Even}(\sigma^*)$ respectively where v_0 is the fixed vertex of σ^* .

Theorem 4.2 : *Let G be a s.c. chordal graph with $p=4n+1$ and a star c.p. σ^* . Let v_0 be the fixed vertex of σ^* . Then*

(i) $\deg_G(v_{ij}) \leq 2n - 1$ for all $v_{ij} \in \text{Odd}(\sigma^*) - \{v_0\}$

(ii) $\deg_G(v_{ij}) \geq 2n + 1$ for all $v_{ij} \in \text{Even}(\sigma^*)$.

Proof: Let $G' = G - v_0$. By Corollary2.2 G' is a s.c. graph with a star c.p. σ^*/σ_1^* . So by Theorem4.1, $\deg_{G'}(v_{ij}) \leq 2n - 1$ for all $v_{ij} \in \text{Odd}(\sigma^*/\sigma_1^*)$ and $\deg_{G'}(v_{ij}) \geq 2n$ for all $v_{ij} \in \text{Even}(\sigma^*/\sigma_1^*)$. Also $\text{Odd}(\sigma^*/\sigma_1^*) = \text{Odd}(\sigma^*) - \{v_0\}$ and $\text{Even}(\sigma^*/\sigma_1^*) = \text{Even}(\sigma^*)$. Hence $\deg_G(v_{ij}) \leq 2n - 1$ for all $v_{ij} \in \text{Odd}(\sigma^*) - \{v_0\}$ and $\deg_G(v_{ij}) \geq 2n + 1$ for all $v_{ij} \in \text{Even}(\sigma^*)$ since by Theorem3.12 $Nhd_G(v_0) = \text{Even}(\sigma^*)$. \square

Let G be a s.c. chordal graph with $4n$ vertices. The graph $G_U \diamond u_0$ obtained from G is also a s.c. chordal graph.

Theorem 4.3 : *Let G be a s.c. chordal graph with $p=4n$. Let*

$U = \{v \in V(G) : \deg_G(v) \geq 2n\}$ and u_0 be a vertex not belonging to $V(G)$. Then $G_U \diamond u_0$ is a s.c. chordal graph.

Proof: By Theorem 2.20 we note that $\omega(G) = 2n$. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ for $1 \leq i \leq s$ be a star c.p. of G . Let E' be the edge set of the graph $G_U \diamond u_0$. Let $\sigma^{*'} = (u_0) \sigma^*$. We prove that $\sigma^{*'}$ is a c.p. of $G_U \diamond u_0$. By the definition of the graph $G_U \diamond u_0$ the pair $[u_0, v_{ij}] \in E'$ if and only if $\deg_G(v_{ij}) \geq 2n$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. By Theorem3.10 $\deg_G(v_{ij}) \geq 2n$ if and only if $\deg_G(\sigma^*(v_{ij})) \leq 2n - 1$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. We note that $\deg_G(\sigma^*(v_{ij})) \leq 2n - 1$ if and only if $[u_0, \sigma^*(v_{ij})] \notin E'$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$ by the definition of the graph $G_U \diamond u_0$. Hence $[u_0, v_{ij}] \in E'$ if and

only if $[u_0, \sigma^*(v_{ij})] \notin E'$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. Therefore $[u_0, v_{ij}] \in E'$ if and only if $[\sigma^{*'}(u_0), \sigma^{*'}(v_{ij})] \notin E'$, since $[\sigma^{*'}(u_0), \sigma^{*'}(v_{ij})] = [u_0, \sigma^{*'}(v_{ij})]$ for all $1 \leq i \leq s$ and $1 \leq j \leq p_i$. For any two distinct vertices v_{ij} and v_{lm} such that $1 \leq i \leq s$, $1 \leq j \leq p_i$, $1 \leq l \leq s$ and $1 \leq m \leq p_l$ the pair $[v_{ij}, v_{lm}] \in E'$ if and only if $[\sigma^{*'}(v_{ij}), \sigma^{*'}(v_{lm})] = [\sigma^*(v_{ij}), \sigma^*(v_{lm})] \notin E'$ since σ^* is a star c.p. of G . Therefore $\sigma^{*'} is a c.p. of $G_U \diamond u_0$. So $G_U \diamond u_0$ is a s.c. graph. By Theorem2.11 $\omega(G_U \diamond u_0) \leq 2n + 1$. We prove that $\omega(G_U \diamond u_0) \geq 2n + 1$. By Theorem4.1 $U = Even(\sigma^*)$. By Theorem2.14 $\langle Even(\sigma^*) \rangle$ is a complete subgraph of G since $\omega(G) = 2n$. Hence $\langle \{u_0\} \cup Even(\sigma^*) \rangle$ is a complete subgraph of $G_U \diamond u_0$. This leads to $\omega(G_U \diamond u_0) \geq 2n + 1$. So $\omega(G_U \diamond u_0) = 2n + 1$. By Theorem2.20 the Theorem follows. $\square$$

Let G be a s.c. chordal graph with $4n+1$ vertices and a c.p. σ . The graph $G - v_0$ obtained from G is also a s.c. chordal graph.

Theorem 4.4 : *Let G be a s.c. chordal graph with $p=4n+1$ and a c.p. σ . Let v_0 be the fixed vertex of σ . Then $G - v_0$ is a s.c. chordal graph.*

Proof: By Corollary2.1 G is a s.c. graph. We prove that $G - v_0$ is chordal by showing $\omega(G - v_0) = 2n$. By Theorem2.11 $\omega(G - v_0) \leq 2n$. By Theorem3.11 $\langle Nhd(v_0) \rangle \cong K_{2n}$. Hence $\omega(G - v_0) \geq 2n$. Then $\omega(G - v_0) = 2n$. The result follows from Theorem2.20. \square

Lemma 4.2 : *Let G be a s.c. chordal graph with $p=4n+1$. Let σ^* be a star c.p. of G with v_0 as its fixed vertex. Let $U = \{v \in V(G - v_0) : deg_{G-v_0}(v) \geq 2n\}$ and u_0 be a vertex not belonging to $V(G - v_0)$. Then $(G - v_0)_U \diamond u_0 \cong G$.*

Proof: Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_1 = (v_0)$ and $\sigma_i = (v_{i1} v_{i2} \cdots v_{ip_i})$ for $2 \leq i \leq s$. By Theorem4.4 $G - v_0$ is a s.c. chordal graph. We note that σ^*/σ_1^* is a star c.p. of $G - v_0$. So by Theorem4.1 $U = Even(\sigma^*/\sigma_1^*)$. By Theorem3.12 $Nhd_G(v_0) = Even(\sigma^*)$. Hence $Nhd_G(v_0) = U$ since $Even(\sigma^*) = Even(\sigma^*/\sigma_1^*)$. Therefore by the definition of the graph $(G - v_0)_U \diamond u_0$ it follows that $(G - v_0)_U \diamond u_0 \cong G$. \square

Lemma 4.3 : *Let G be a s.c. chordal graph with $p=4n$ and a star c.p. σ^* . Let $U = \{v \in V(G) : deg_G(v) \geq 2n\}$ and u_0 be a vertex not belonging to $V(G)$. Let $\sigma^{*'} be a star c.p. of $G_U \diamond u_0$ with v'_0 as its fixed vertex. Then $(G_U \diamond u_0) - v'_0 \cong G$.$*

Proof: By Theorem 4.1 $U = \text{Even}(\sigma^*)$. So $Nhd_{G_U \diamond u_0}(u_0) = \text{Even}(\sigma^*)$. Therefore $|Nhd_{G_U \diamond u_0}(u_0)| = 2n$ since $|\text{Even}(\sigma^*)| = 2n$. Then $\deg_{G_U \diamond u_0}(u_0) = 2n$. Hence by Theorem 4.2 $u_0 = v'_0$. Therefore $(G_U \diamond u_0) - v'_0 \cong G$. \square

The following result states that the verification of whether two given s.c. chordal graphs with $4n$ vertices are isomorphic or not could be equivalently posed as a problem of verifying whether two given s.c. chordal graphs with $4n+1$ vertices are isomorphic or not.

Theorem 4.5 : *Let G' and G'' be s.c. chordal graphs with $4n$ vertices. Let*

$U' = \{v \in V(G') : \deg_{G'}(v) \geq 2n\}$ and $U'' = \{v \in V(G'') : \deg_{G''}(v) \geq 2n\}$. Let u'_0 and u''_0 be vertices not belonging to $V(G')$ and $V(G'')$ respectively. Then

(i) $G'_{U'} \diamond u'_0$ and $G''_{U''} \diamond u''_0$ are s.c. chordal graphs with $4n+1$ vertices.

(ii) $G' \cong G''$ if and only if $G'_{U'} \diamond u'_0 \cong G''_{U''} \diamond u''_0$.

Proof: (i) Follows from Theorem 4.3.

(ii) Let $G' \cong G''$. Let ψ be a vertex isomorphism of G' onto G'' . We note that $\psi(U') = U''$.

We define a 1-1 correspondence $\tilde{\psi}$ from $V(G'_{U'} \diamond u'_0)$ onto $V(G''_{U''} \diamond u''_0)$ as follows.

$$\tilde{\psi}(v) = \begin{cases} \psi(v) & \text{if } v \in V(G') \\ u''_0 & \text{if } v = u'_0 \end{cases}$$

We prove that $\tilde{\psi}$ is a vertex isomorphism of $G'_{U'} \diamond u'_0$ onto $G''_{U''} \diamond u''_0$. Let $E' = E(G'_{U'} \diamond u'_0)$ and $E'' = E(G''_{U''} \diamond u''_0)$. Let v be a vertex of $G'_{U'} \diamond u'_0$ distinct from u'_0 . By the definition of the graph $G'_{U'} \diamond u'_0$ we note that $[u'_0, v] \in E'$ if and only if $v \in U'$. By the definition of the graph $G''_{U''} \diamond u''_0$ the pair $[u''_0, \psi(v)] \in E''$ if and only if $\psi(v) \in U''$. Hence $[u'_0, v] \in E'$ if and only if $[\tilde{\psi}(u'_0), \tilde{\psi}(v)] = [u''_0, \psi(v)] \in E''$ since $\psi(U') = U''$. Let u and v be two vertices of $G'_{U'} \diamond u'_0$ distinct from u'_0 . We note that both u and v are vertices of G' . Hence $[u, v] \in E'$ if and only if $[\tilde{\psi}(u), \tilde{\psi}(v)] = [\psi(u), \psi(v)] \in E''$ since ψ is a vertex isomorphism of G' onto G'' . Therefore $\tilde{\psi}$ is a vertex isomorphism of $G'_{U'} \diamond u'_0$ onto $G''_{U''} \diamond u''_0$. Hence $G'_{U'} \diamond u'_0 \cong G''_{U''} \diamond u''_0$.

Let $G'_{U'} \diamond u'_0 \cong G''_{U''} \diamond u''_0$. Let ψ be a vertex isomorphism of $G'_{U'} \diamond u'_0$ onto $G''_{U''} \diamond u''_0$. By (i) of this Theorem $G'_{U'} \diamond u'_0$ and $G''_{U''} \diamond u''_0$ are s.c. chordal graphs with $4n+1$ vertices. Let $\sigma^{*'} and $\sigma^{*''}$ be star c.p.'s of $G'_{U'} \diamond u'_0$ and $G''_{U''} \diamond u''_0$ respectively. Let v'_0 and v''_0 be the fixed vertices of $\sigma^{*'}$ and $\sigma^{*''}$ respectively. By Corollary 3.1 $\deg_{G'_{U'} \diamond u'_0}(v'_0) = 2n$. Then$

$\deg_{G''_{U''} \diamond u''_0}(\psi(v'_0)) = \deg_{G'_{U'} \diamond u'_0}(v'_0) = 2n$. Hence by Theorem 4.2 $\psi(v'_0) = v''_0$. By Lemma 4.1 $G'_{U'} \diamond u'_0 - v'_0 \cong G''_{U''} \diamond u''_0 - \psi(v'_0)$. So $(G'_{U'} \diamond u'_0) - v'_0 \cong (G''_{U''} \diamond u''_0) - v''_0$. By Lemma 4.3 $(G'_{U'} \diamond u'_0) - v'_0 \cong G'$ and $(G''_{U''} \diamond u''_0) - v''_0 \cong G''$. This leads to $G' \cong G''$. \square

The following result states that the verification of whether two given s.c. chordal graphs with $4n+1$ vertices are isomorphic or not could be equivalently posed as a problem of verifying whether two given s.c. chordal graphs with $4n$ vertices are isomorphic or not.

Theorem 4.6 : *Let G' and G'' be s.c. chordal graphs with $4n+1$ vertices. Let $\sigma^{*'} and $\sigma^{*''}$ be star c.p. of G' and G'' respectively. Let v'_0 and v''_0 be the fixed vertices of $\sigma^{*'}$ and $\sigma^{*''}$ respectively. Then$*

- (i) $\deg_{G'}(v'_0) = 2n$ and $\deg_{G'}(v) \neq 2n$ for all vertices v of G' distinct from v'_0
- (ii) $\deg_{G''}(v''_0) = 2n$ and $\deg_{G''}(v) \neq 2n$ for all vertices v of G'' distinct from v''_0
- (iii) $G' - v'_0$ and $G'' - v''_0$ are s.c. chordal graphs with $4n$ vertices
- (iv) $G' \cong G''$ if and only if $G' - v'_0 \cong G'' - v''_0$.

Proof: (i) and (ii) follow from Corollary 3.1 and Theorem 4.2.

(iii) Follows from Theorem 4.4.

(iv) Let $U' = \{v \in V(G' - v'_0) : \deg_{G' - v'_0}(v) \geq 2n\}$ and $U'' = \{v \in V(G'' - v''_0) : \deg_{G'' - v''_0}(v) \geq 2n\}$. Let u'_0 and u''_0 be vertices not belonging to $V(G' - v'_0)$ and $V(G'' - v''_0)$ respectively. By Theorem 4.5(ii) $G' - v'_0 \cong G'' - v''_0$ if and only if $(G' - v'_0)_{U'} \diamond u'_0 \cong (G'' - v''_0)_{U''} \diamond u''_0$. By Lemma 4.2 $(G' - v'_0)_{U'} \diamond u'_0 \cong G'$ and $(G'' - v''_0)_{U''} \diamond u''_0 \cong G''$. Hence $G' \cong G''$ if and only if $G' - v'_0 \cong G'' - v''_0$. \square

The following algorithm constructs a s.c. chordal graph with $4n+1$ vertices from a s.c. chordal graph with $4n$ vertices.

Algorithm 4.1 : Given a s.c. chordal graph with $4n$ vertices this algorithm constructs a s.c. chordal graph with $4n+1$ vertices

Input : A s.c. chordal graph with $4n$ vertices.

Step 1 : Locate all the vertices v such that $\deg_G(v) \geq 2n$. Call the set of all such vertices as U .

Output : *The graph $G_U \diamond u_0$ where u_0 is a vertex distinct from the vertices of G .*

The following result justifies the claim of Algorithm4.1.

Theorem 4.7 : *Let G be a s.c. chordal graph with $4n$ vertices. The output graph $G_U \diamond u_0$ obtained by Algorithm4.1 with G as its input graph is a s.c. chordal graph with $4n+1$ vertices.*

Proof: Follows from Theorem4.3. \square

The complexity of Algorithm4.1 is given by the following result.

Theorem 4.8 : *Algorithm4.1 is computable in linear time.*

Proof: Let a s.c. chordal graph G with $4n$ vertices be an input to Algorithm4.1. We note that Step1 of Algorithm4.1 takes $O(n)$ time for the input G since it has to examine the degrees of the $4n$ vertices of G to locate all the vertices of degree atleast $2n$. So Algorithm4.1 is computable in linear time. \square

To illustrate Algorithm4.1 we consider the s.c. chordal graph G with 8 vertices shown in Figure 4.1(a). Let G be the input to Algorithm4.1. Step 1 of Algorithm4.1 locates all the vertices v of G such that $\deg_G(v) \geq 4$. We note $U = \{2, 4, 6, 8\}$. The output $G_U \diamond u_0$ is shown in Figure 4.1(b).

The following algorithm constructs a s.c. chordal graph with $4n$ vertices from a s.c. chordal graph with $4n+1$ vertices.

Algorithm 4.2 : *Given a s.c. chordal graph with $4n+1$ vertices this algorithm constructs a s.c. chordal graph with $4n$ vertices.*

Input : *A s.c. chordal graph G with $4n+1$ vertices.*

Step 1 : *Locate the vertex v of G such that $\deg_G(v) = 2n$. Call that vertex as v_0 .*

Output : *The graph $G - v_0$.*

The following result justifies the claim of Algorithm4.2.

Theorem 4.9 : *Let G be s.c. chordal graph with $4n+1$ vertices. The output graph $G - v_0$ obtained by Algorithm4.2 with G as its input graph is a s.c. chordal graph with $4n$ vertices.*

Proof: Follows from Corollary3.1, Theorem4.2 and Theorem4.4. \square

The complexity of Algorithm4.2 is given by the following result.

Theorem 4.10 : *Algorithm4.2 is computable in linear time.*

Proof: Let a s.c. chordal graph G with $4n+1$ vertices be an input to Algorithm4.2. We note that Step1 of Algorithm4.2 takes $O(n)$ time for the input G since it has to examine the degrees of atmost $4n+1$ vertices of G to locate the vertex of degree $2n$ in G . Hence Algorithm4.2 is computable in linear time. \square

To illustrate Algorithm4.2 we consider the s.c. chordal graph G with 9 vertices shown in Figure 4.2(a). Let G be the input to Algorithm4.2. Step 1 of Algorithm4.2 locates the vertex 9 which is the vertex of degree 4 in G . The output graph $G - v_0$ is shown in Figure4.2(b) where $v_0 = 9$.

Isomorphism of s.c. chordal graphs is algorithmically as hard as the isomorphism of some of its subclasses.

Theorem 4.11 : *The isomorphism of the following classes of graphs are polynomially equivalent.*

- (i) *S.c. chordal graphs.*
- (ii) *S.c. chordal graphs with $4n$ vertices where n is a positive integer.*
- (iii) *S.c. chordal graphs with $4n+1$ vertices where n is a positive integer.*

Proof: We prove that the isomorphism of (ii) is polynomially equivalent to the isomorphism of (iii).

Let G' and G'' be two s.c. chordal graphs with $4n$ vertices. Let $U' = \{v \in V(G') : \deg_{G'}(v) \geq 2n\}$ and $U'' = \{v \in V(G'') : \deg_{G''}(v) \geq 2n\}$. Let u'_0 and u''_0 be vertices not belonging to $V(G')$ and $V(G'')$ respectively. Construct the graphs $G'_{U'} \diamond u'_0$ and $G''_{U''} \diamond u''_0$ by using Algorithm4.1. By Theorem4.7 the graphs $G'_{U'} \diamond u'_0$ and $G''_{U''} \diamond u''_0$ are s.c. chordal graphs with $4n+1$ vertices. By Theorem4.5 G' is isomorphic to G'' if and only if $G'_{U'} \diamond u'_0$ is isomorphic to $G''_{U''} \diamond u''_0$. Hence the isomorphism of (ii) is polynomially reducible to the isomorphism of (iii) since by Theorem4.8 Algorithm4.1 is polynomially computable.

Let G' and G'' be two s.c. chordal graphs with $4n+1$ vertices. Construct the graphs $G' - v'_0$ and $G'' - v''_0$ by using Algorithm4.2 where v'_0 and v''_0 are the vertices of G' and G'' such that $\deg_{G'}(v'_0) = 2n$ and $\deg_{G''}(v''_0) = 2n$. By Theorem4.9 the graphs $G' - v'_0$ and $G'' - v''_0$ are s.c. chordal graphs with $4n$ vertices. By Theorem4.6 G' is isomorphic to G'' if and only if $G' - v'_0$ is isomorphic to $G'' - v''_0$. Hence the isomorphism of (iii) is polynomially reducible to the isomorphism of (ii) since by Theorem4.10 Algorithm4.2 is polynomially computable.

Hence the isomorphism of (ii) is polynomially equivalent to the isomorphism of (iii).

Therefore the Theorem follows from the fact that every s.c. chordal graph has $4n$ or $4n+1$ vertices. \square

4.3 On the catalogue compilation of self-complementary chordal graphs

The following algorithm recognises whether a given s.c. graph with $4n$ vertices is chordal or not.

Algorithm 4.3 : Given a s.c. graph G with $4n$ vertices this algorithm decides whether G is chordal or not.

Input : A s.c. graph G with $4n$ vertices.

Step 1 : Compute the degrees of the vertices of G .

Step 2 : Arrange the degrees of the vertices of G in non-increasing order. Let

$d_1 \geq d_2 \geq \dots \geq d_{4n}$ be the sequence of the degrees of the vertices of G arranged in non-increasing order.

Step 3 : Compute the sum $\sum_{i=1}^{2n} d_i$.

Output : Whether the graph G is chordal or not is decided as follows. The graph G is chordal if $\sum_{i=1}^{2n} d_i = 6n^2 - 2n$ else it is not chordal.

The following result justifies the claim of Algorithm4.3.

Theorem 4.12 : *Let G be a s.c. graph with $4n$ vertices where n is a positive integer. Then G is a chordal graph if and only if it is decided as a chordal graph by Algorithm4.3.*

Proof: Follows from Theorem2.34 (i). \square

The complexity of Algorithm4.3 is given by the following result.

Theorem 4.13 : *Let G be a s.c. graph with $4n$ vertices where n is a positive integer. Algorithm4.3 with G as its input graph decides whether G is chordal or not in $O(n)$ time.*

Proof: Step1 of Algorithm4.3 takes $O(n)$ time since all the $4n$ vertices of G should be examined to compute the degrees of the vertices of G . The degrees of the vertices of G are integers and take values between 0 and $4n - 1$. So it takes $O(n)$ time for sorting the degrees of the $4n$ vertices of G [1]. Step3 of Algorithm4.3 takes $O(1)$ time. Therefore the result. \square

To illustrate Algorithm4.3 we consider the graphs G' and G'' shown in Figure 4.3(a) and Figure 4.3(b) respectively.

Let G' be the graph shown in Figure 4.3(a). The graph G' is a s.c. graph since (1 2 3 4 5 6 7 8 9 10 11 12) is a c.p. of G' . Let G' be the input to Algorithm4.3. Step 1 of Algorithm4.3 yields $\deg_{G'}(2) = \deg_{G'}(4) = \deg_{G'}(6) = \deg_{G'}(8) = \deg_{G'}(10) = \deg_{G'}(12) = 8$ and $\deg_{G'}(1) = \deg_{G'}(3) = \deg_{G'}(5) = \deg_{G'}(7) = \deg_{G'}(9) = \deg_{G'}(11) = 3$. Step 2 of Algorithm4.3 arranges the degrees of the vertices of G' in non-increasing order $d_1 \geq d_2 \geq \dots \geq d_{12}$ where $d_i = 8$ for $1 \leq i \leq 6$ and $d_i = 3$ for $7 \leq i \leq 12$. By Step 3 of Algorithm4.3 $\sum_{i=1}^6 d_i = 48$. Since $6n^2 - 2n = 48 = \sum_{i=1}^6 d_i$ Algorithm4.3 decides the graph G' is a chordal graph.

Let G'' be the graph shown in Figure 4.3(b). The graph G'' is s.c. since (1 2 3 4 5 6 7 8) is a c.p. of G'' . Let G'' be the input to Algorithm4.3. Step 1 of Algorithm4.3 yields $\deg_{G''}(2) = \deg_{G''}(4) = \deg_{G''}(6) = \deg_{G''}(8) = 4$ and $\deg_{G''}(1) = \deg_{G''}(3) = \deg_{G''}(5) = \deg_{G''}(7) = 3$. Step 2 of Algorithm4.3 arranges the degrees of the vertices of G'' in non-increasing order $d_1 \geq d_2 \geq \dots \geq d_8$ where $d_i = 4$ for $1 \leq i \leq 4$ and $d_i = 3$ for $5 \leq i \leq 8$. By Step 3 of Algorithm4.3 $\sum_{i=1}^4 d_i = 16$. Since $6n^2 - 2n = 20 \neq \sum_{i=1}^4 d_i$ Algorithm4.3 decides the graph G'' is not a chordal graph.

The following algorithm recognises whether a given s.c. graph with $4n+1$ vertices is chordal or not.

Algorithm 4.4 : Given a s.c. graph G with $4n+1$ vertices this algorithm decides whether G is chordal or not.

Input : A s.c. graph G with $4n+1$ vertices.

Step 1 : Compute the degrees of the vertices of G .

Step 2 : Arrange the degrees of the vertices of G in non-increasing order. Let

$d_1 \geq d_2 \geq \dots \geq d_{4n+1}$ be the sequence of the degrees of the vertices of G arranged in non-increasing order.

Step 3 : Compute the sum $\sum_{i=1}^{2n} d_i$.

Output : Whether the graph G is chordal or not is decided as follows. The graph G is chordal if $\sum_{i=1}^{2n} d_i = 6n^2$ else it is not chordal.

The following result justifies the claim of Algorithm4.4.

Theorem 4.14 : Let G be a s.c. graph with $4n+1$ vertices where n is a positive integer. Then G is a chordal graph if and only if it is decided as a chordal graph by Algorithm4.4.

Proof: Follows from Theorem2.34 (ii). \square

The complexity of Algorithm4.4 is given by the following result.

Theorem 4.15 : Let G be a s.c. graph with $4n+1$ vertices where n is a positive integer. Algorithm4.4 with G as its input graph decides whether G is chordal or not in $O(n)$ time.

Proof: Step1 of Algorithm4.4 takes $O(n)$ time since all the $4n+1$ vertices of G should be examined to compute the degrees of the vertices of G . The degrees of the vertices of G are integers and take values between 0 and $4n$. So it takes $O(n)$ time for sorting the degrees of the $4n+1$ vertices of G [1]. Step3 of Algorithm4.4 takes $O(1)$ time. Therefore the result. \square

To illustrate Algorithm4.4 we consider the graphs G' and G'' shown in Figure 4.4(a) and Figure 4.4(b) respectively.

Let G' be the graph shown in Figure 4.4(a). The graph G' is s.c. since $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)(13)$ is a c.p. of G' . Let G' be the input to Algorithm4.4. Step 1 of Algorithm4.4 yields $\deg_{G'}(2) = \deg_{G'}(4) = \deg_{G'}(6) = \deg_{G'}(8) = \deg_{G'}(10) = \deg_{G'}(12) = 9$ and $\deg_{G'}(1) = \deg_{G'}(3) = \deg_{G'}(5) = \deg_{G'}(7) = \deg_{G'}(9) = \deg_{G'}(11) = 3$ and $\deg_{G'}(13) = 6$. Step 2 of Algorithm4.4 arranges the degrees of the vertices of G' in non-increasing order $d_1 \geq d_2 \geq \dots \geq d_{13}$ where $d_i = 9$ for $1 \leq i \leq 6$, $d_7 = 6$ and $d_i = 3$ for $8 \leq i \leq 13$. By Step 3 of Algorithm4.4 $\sum_{i=1}^6 d_i = 54$. Since $6n^2 = 54 = \sum_{i=1}^6 d_i$ Algorithm4.4 decides the graph G' is a chordal graph.

Let G'' be the graph shown in Figure 4.4(b). The graph G'' is s.c. since $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9)$ is a c.p. of G'' . Let G'' be the input to Algorithm4.4. Step 1 of Algorithm4.4 yields $\deg_{G''}(2) = \deg_{G''}(4) = \deg_{G''}(6) = \deg_{G''}(8) = 5$, $\deg_{G''}(1) = \deg_{G''}(3) = \deg_{G''}(5) = \deg_{G''}(7) = 3$ and $\deg_{G''}(9) = 4$. Step 2 of Algorithm4.4 arranges the degrees of the vertices of G'' in non-increasing order $d_1 \geq d_2 \geq \dots \geq d_9$ where $d_i = 4$ for $1 \leq i \leq 4$, $d_5 = 4$ and $d_i = 3$ for $6 \leq i \leq 9$. By Step 3 of Algorithm4.4 $\sum_{i=1}^4 d_i = 16$. Since $6n^2 = 24 \neq \sum_{i=1}^4 d_i$ Algorithm4.4 decides the graph G'' is not a chordal graph.

Remark: A graph can be recognised to be chordal or not in linear time using the algorithms given in [131], [184] and [186]. However recognising whether a chordal graph is self-complementary or not appears to be difficult.

Using Algorithm4.3 and Algorithm4.4 we compile the catalogue of s.c. chordal graphs with atmost 12 vertices from the available catalogue of s.c. graphs with atmost 12 vertices.

There is only one non-isomorphic s.c. chordal graph with 4 vertices. This graph is isomorphic to the graph with the following 4×4 matrix as its adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

There is only one non-isomorphic s.c. chordal graph with 5 vertices. This graph is isomorphic to the graph with the following 5×5 matrix as its adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

There are 3 non-isomorphic s.c. chordal graphs with 8 vertices. They are isomorphic to the graphs with the following 8×8 matrices as its adjacency matrices.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

There are 3 non-isomorphic s.c. chordal graphs with 9 vertices. They are isomorphic to the graphs with the following 9×9 matrices as its adjacency matrices.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

There are 16 non-isomorphic s.c. chordal graphs with 12 vertices. They are isomorphic to the graphs with the following 12×12 matrices as its adjacency matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

[illegible]

Remark : We note that the catalogue of s.c. chordal graphs with p vertices can be compiled from the catalogue of s.c. graphs with p vertices for $p > 12$ by following the same method adopted for compiling the catalogue of s.c. chordal graphs with p vertices for $p \leq 12$. By Theorem4.13 and Theorem4.15 deciding whether a s.c. graph with $4n$ or $4n+1$ vertices is chordal or not takes $O(n)$ time. Scanning the entire list of non-isomorphic s.c. graphs with $4n$ or $4n+1$ vertices takes $O(s_{4n})$ time or $O(s_{4n+1})$ time respectively. So the running time of the algorithm which locates the non-isomorphic s.c. chordal graphs with p vertices from the list of all non-isomorphic s.c. graphs with p vertices takes $O(ns_{4n})$ time or $O(ns_{4n+1})$ time accordingly as $p=4n$ or $p=4n+1$ respectively.

The following result gives a method for constructing all s.c. chordal graphs with $4n+1$ vertices from the set of all s.c. chordal graphs with $4n$ vertices.

Theorem 4.16 : *Let \mathcal{G} be the set of all non-isomorphic s.c. chordal graphs with $4n$ vertices.*

Let $\mathcal{G}^ = \{G_U \diamond u_0 : G \in \mathcal{G}, u_0 \notin V(G) \text{ and } U = \{v \in V(G) : \deg_G(v) \geq 2n\}\}$.*

Then \mathcal{G}^ is the set of all non-isomorphic s.c. chordal graphs with $4n+1$ vertices.*

Proof: Let G' and G'' be two s.c. chordal graphs with $4n$ vertices. Let u'_0 and u''_0 be such that $u'_0 \notin V(G')$ and $u''_0 \notin V(G'')$. Let $U' = \{v \in V(G') : \deg_{G'}(v) \geq 2n\}$ and $U'' = \{v \in V(G'') : \deg_{G''}(v) \geq 2n\}$. By Theorem4.5 $G' \cong G''$ if and only if $G'_{U'} \diamond u'_0 \cong G''_{U''} \diamond u''_0$. Hence the Theorem. \square

Using Theorem4.16 we obtain the catalogue of s.c. chordal graphs with 13 vertices from the set of all non-isomorphic s.c. chordal graphs with 12 vertices.

There are 16 non-isomorphic s.c. chordal graphs with 13 vertices. They are isomorphic to the graphs with the following 13×13 matrices as its adjacency matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Remark : By using Theorem 4.16 the catalogue of s.c. chordal graphs with $4n+1$ vertices can be compiled from the catalogue of s.c. chordal graphs with $4n$ vertices where $n > 3$ as we have done for the case of s.c. chordal graphs with 13 vertices. We note that the complexity of the algorithm which constructs all non-isomorphic s.c. chordal graphs with $4n+1$ vertices from all non-isomorphic s.c. chordal graphs with $4n$ vertices by using the method followed above takes $O(n\zeta_{4n})$ where ζ_{4n} is the number of non-isomorphic s.c. chordal graphs with $4n$ vertices.

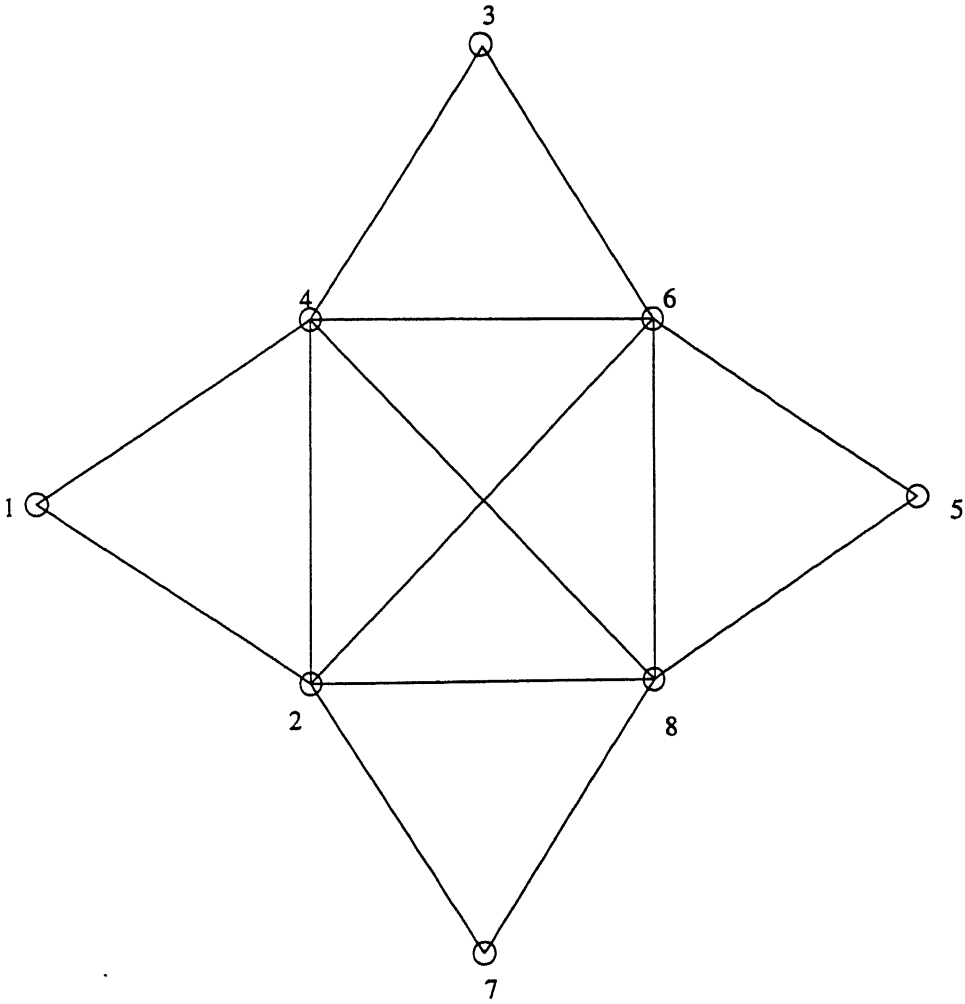


Figure 4.1a

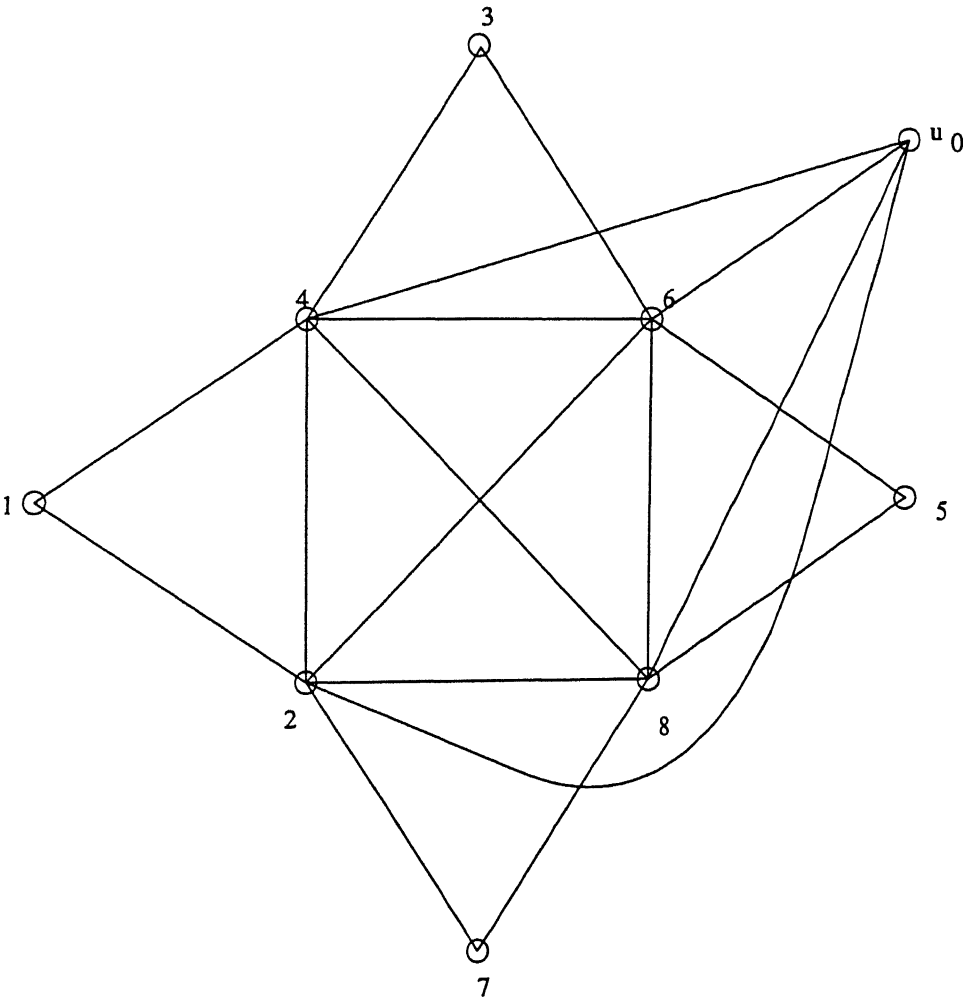


Figure 4.1b

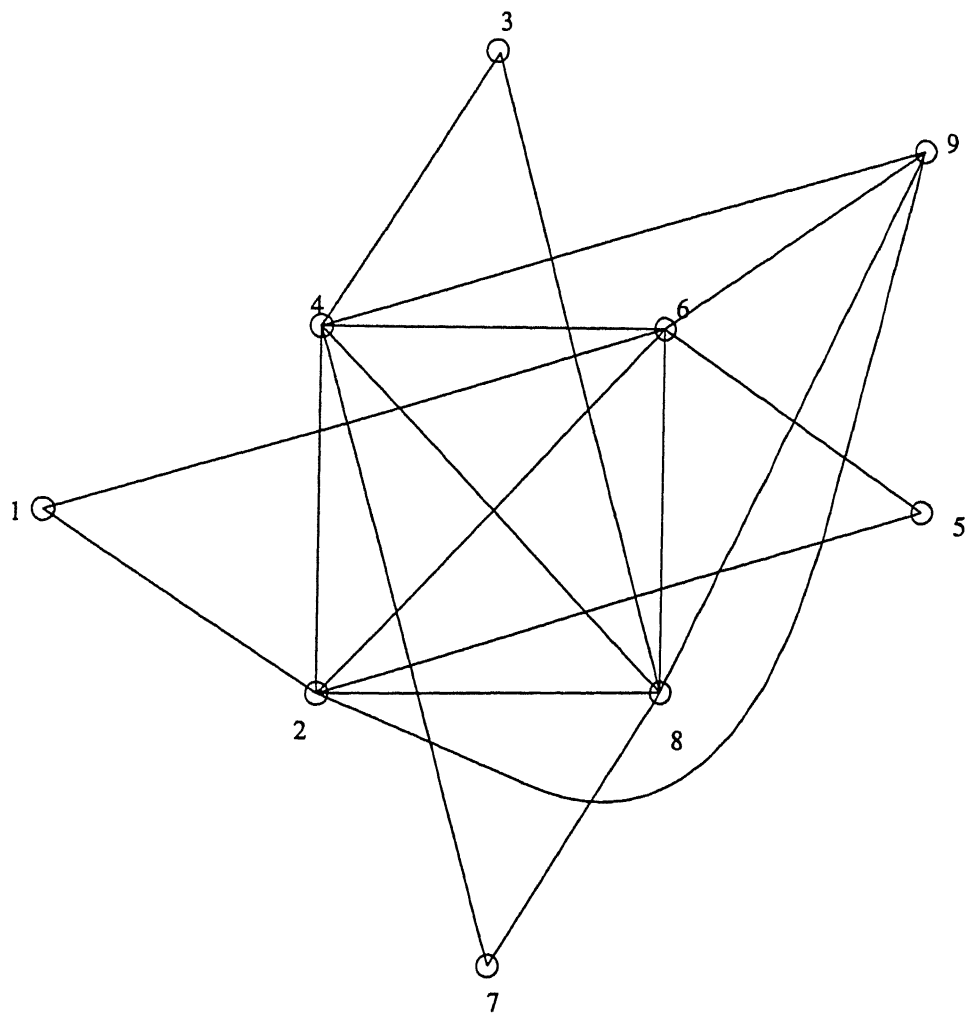


Figure 4.2a

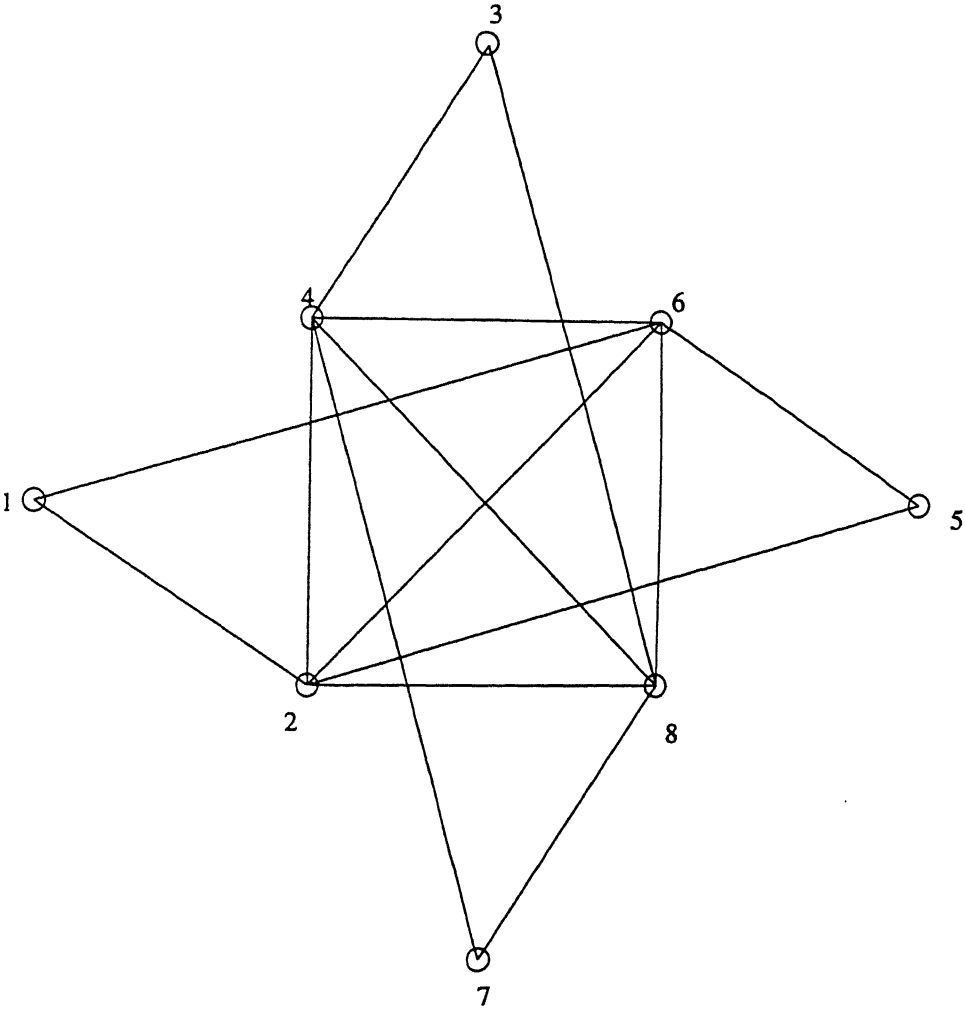


Figure 4.2b

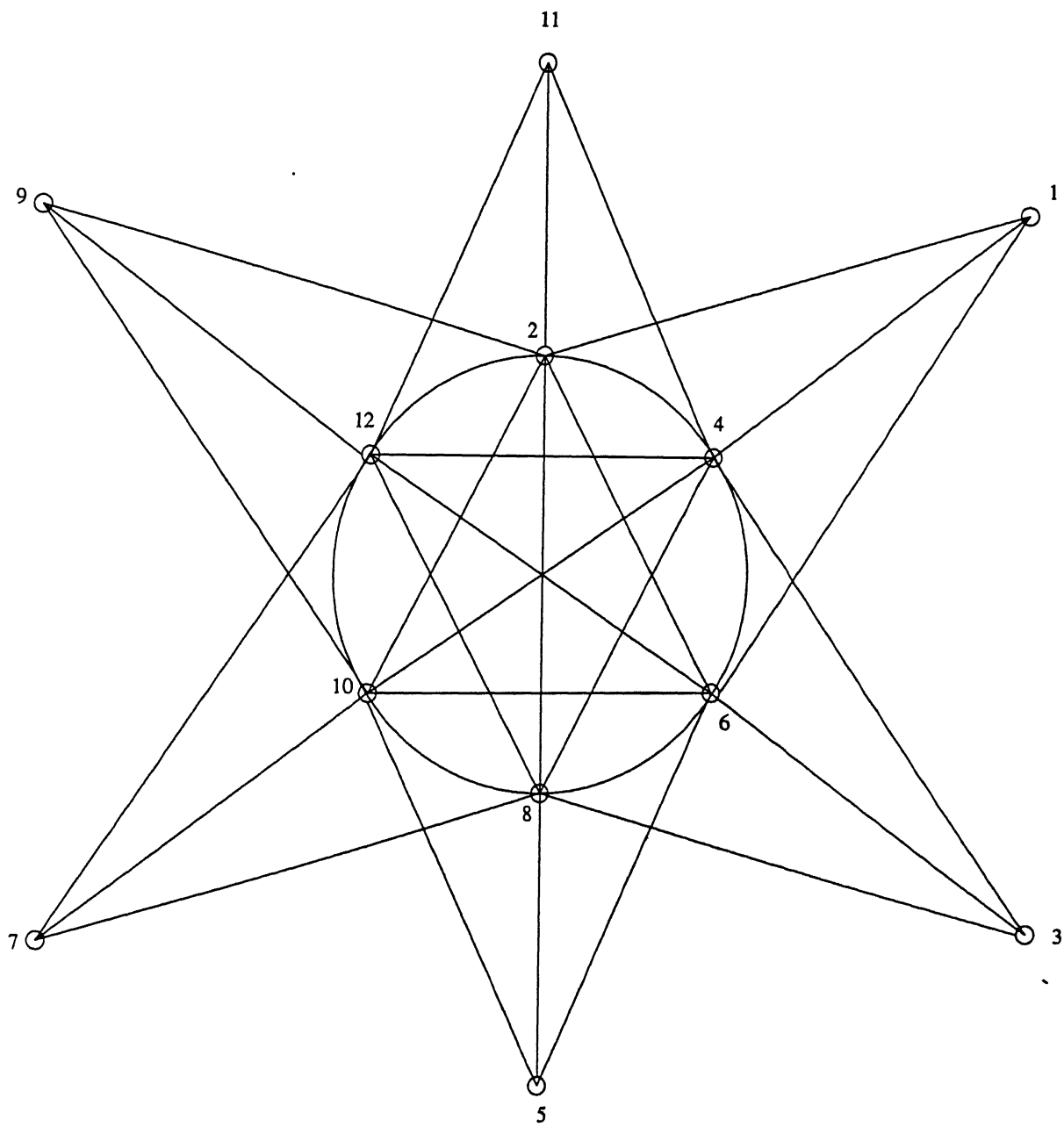


Figure 4.3a

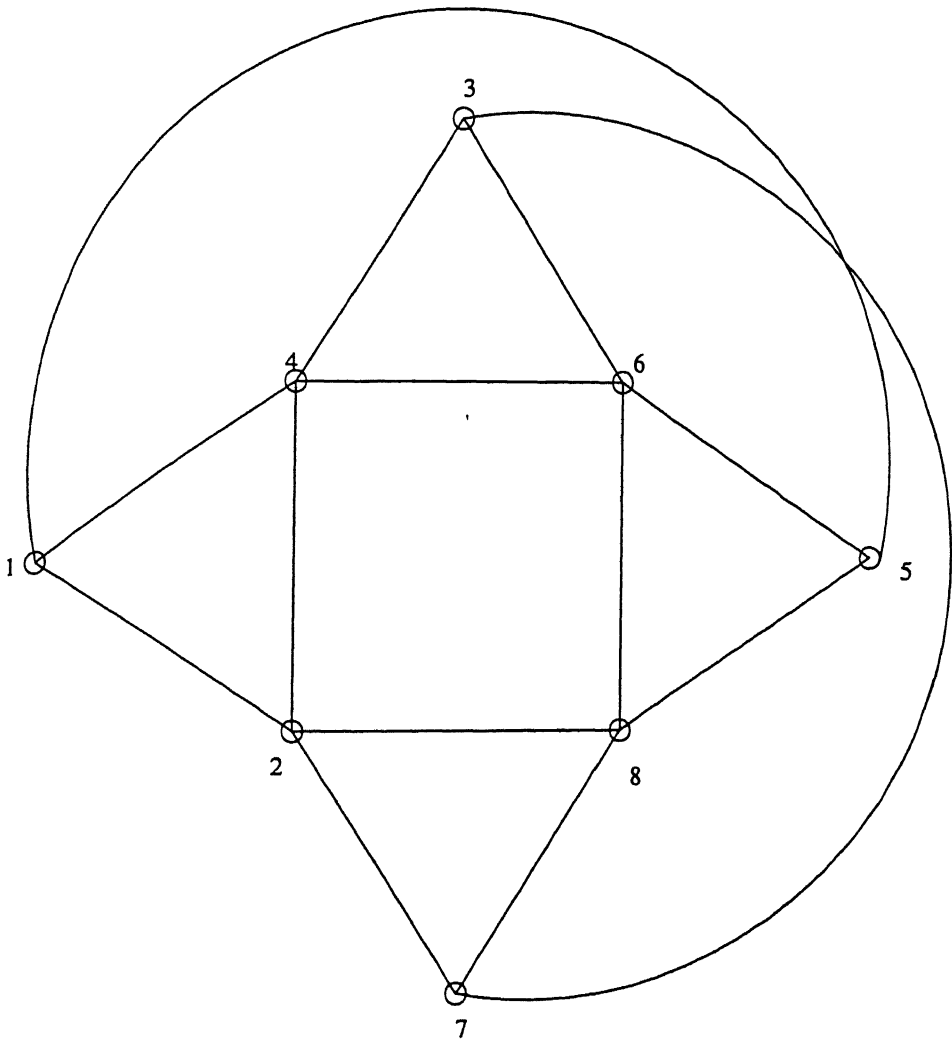


Figure 4.3b

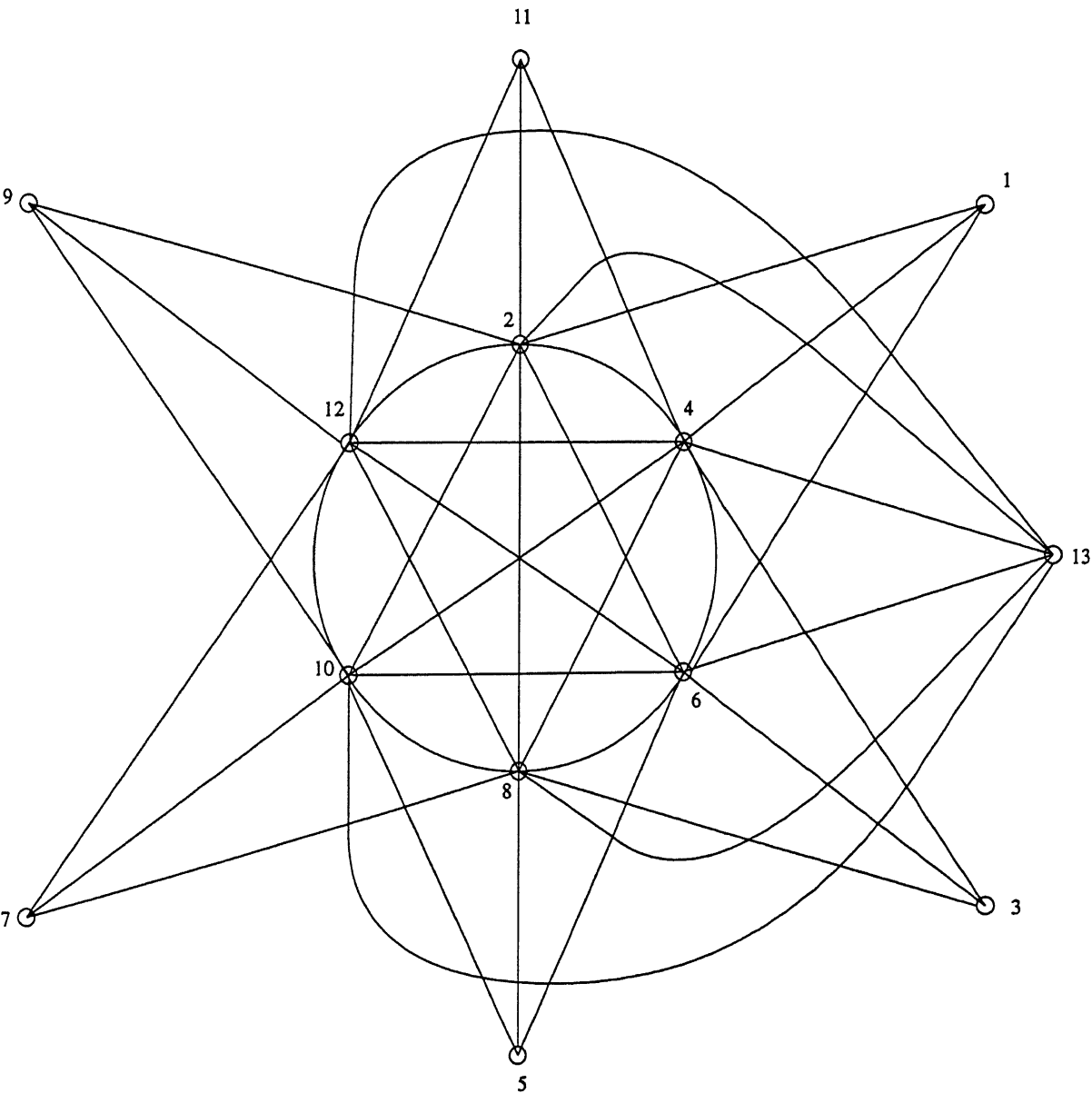


Figure 4.4a

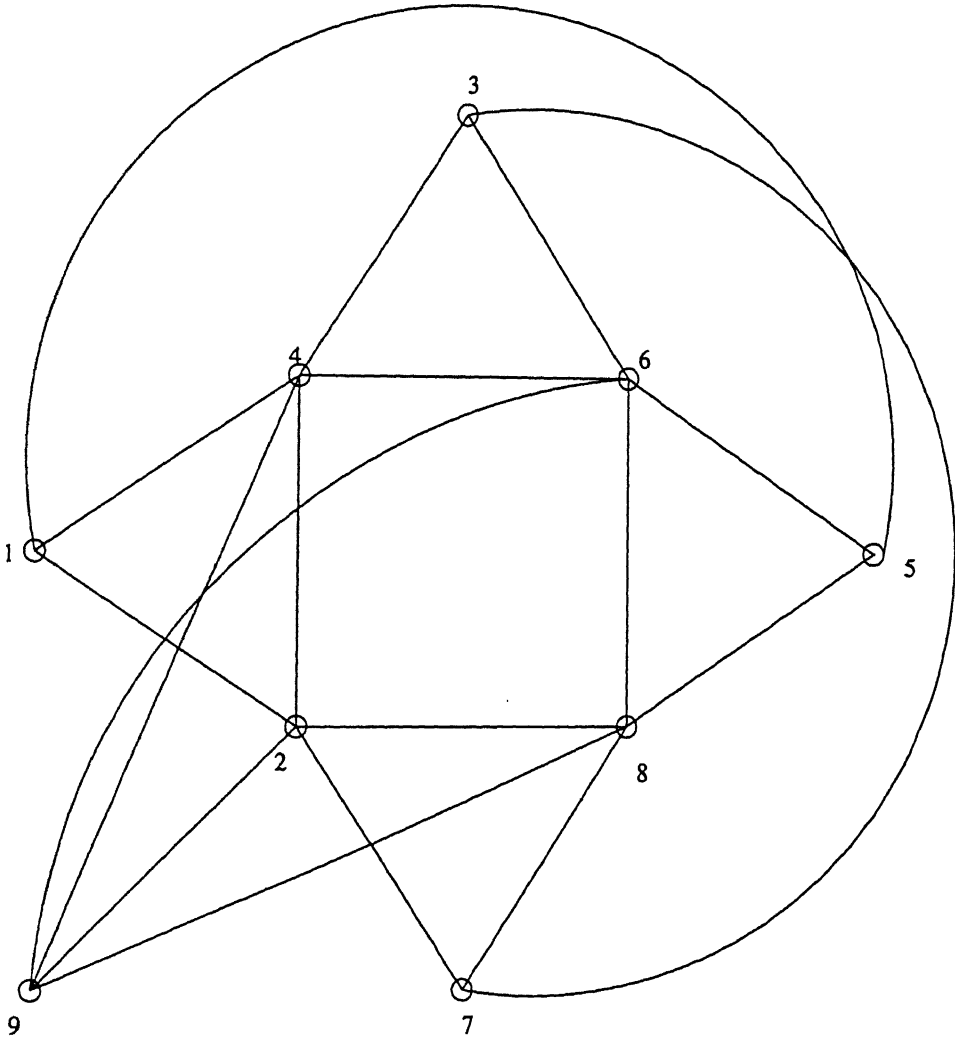


Figure 4.4b

Chapter 5

Self-complementary chordal graphs and some graph parameters

5.1 Introduction

Graph parameters have been studied because of the usefulness in determining the structure of the graph.

In this chapter we study about the following graph parameters for s.c. chordal graphs.

- (a) Chromatic number.
- (b) Chromatic index.
- (c) Domination number.
- (d) Spectrum.

In general obtaining the exact value of the chromatic number of a graph is quite difficult. However researchers had obtained bounds for the chromatic number of graphs and several classes of graphs (see [122] and [217]). In Section 5.2 we prove that the chromatic number of s.c. chordal graphs with $4n$ vertices is $2n$ and that of s.c. chordal graphs with $4n+1$ vertices is $2n+1$. We also obtain bounds on the chromatic number of s.c. perfect graphs and prove that the upper bounds are attained if and only if the graph is s.c. chordal.

Vizing [226] proved that the chromatic index of a graph G is $\Delta(G)$ or $\Delta(G) + 1$. Since then research has progressed much in classifying the graphs with chromatic index $\Delta(G)$ (Class 1 graphs) and the graphs with chromatic index $\Delta(G) + 1$ (Class 2 graphs) [81], [82] and [83]. In Section 5.3 we give a sufficient condition for a s.c. chordal graph with $4n$ vertices to be a Class 1 graph and establish the Class 1 property for some classes of s.c. chordal graphs with $4n$ vertices. We also obtain bounds for the chromatic index of s.c. chordal graphs.

Results on the domination number of graphs and several classes of graphs have been surveyed in [31], [62], [116] and [117]. Booth in [31] has proved that computing domination number of chordal graphs is NP-Complete. However linear time algorithm exists for computing independent domination number of chordal graphs [79]. In Section 5.4 we prove that the domination number and the independent domination number can be atmost $2n$ for s.c. chordal graphs with $4n$ vertices and atmost $2n+1$ for s.c. chordal graphs with $4n+1$ vertices.

The spectrum of a graph provides a wealth of information about the graph though by no means it specifies the graph uniquely [193]. For further reading on this topic we refer to [72] and [193]. Existence of cospectral graphs with several restrictions had been studied by researchers [72] and [193]. In Section 5.5 we prove that the least positive integer for which there exists cospectral s.c. chordal graphs is 12. We also obtain bounds for the maximum eigen-value of a s.c. chordal graph.

5.2 On the chromatic number of self-complementary chordal graphs

The following result on the chromatic number of a graph and its complement was obtained by Nordhaus and Gaddum [145].

Theorem 5.1 (Nordhaus and Gaddum [145]) : *For any graph G with p vertices the following holds.*

$$(i) \quad \lceil 2\sqrt{p} \rceil \leq \chi(G) + \chi(\bar{G}) \leq p + 1$$

$$(ii) \ p \leq \chi(G)\chi(\bar{G}) \leq \lfloor (\frac{p+1}{2})^2 \rfloor .$$

The following result gives bounds for the chromatic number of a s.c. graph.

Theorem 5.2 : *Let G be a s.c. graph with p vertices. Then*

$$(i) \ \lceil \sqrt{4n} \rceil \leq \chi(G) \leq 2n \text{ when } p=4n$$

$$(ii) \ \lceil \sqrt{4n+1} \rceil \leq \chi(G) \leq 2n+1 \text{ when } p=4n+1.$$

Proof: (i) Let G be a s.c. graph with $4n$ vertices. We note that $\chi(G) = \chi(\bar{G})$ since G is a s.c. graph. So by Theorem 5.1 $\chi(G)$ should satisfy the inequations given by (5.1).

$$\left. \begin{aligned} \lceil 2\sqrt{4n} \rceil &\leq 2\chi(G) \leq 4n+1 \\ 4n &\leq (\chi(G))^2 \leq \lfloor (\frac{4n+1}{2})^2 \rfloor \end{aligned} \right\} \quad (5.1)$$

Then it follows that

$$\lceil \sqrt{4n} \rceil \leq \chi(G) \leq 2n .$$

(ii) Let G be a s.c. graph with $4n+1$ vertices. We note that $\chi(G) = \chi(\bar{G})$ since G is a s.c. graph. So by Theorem 5.1 $\chi(G)$ should satisfy the inequations given by (5.2).

$$\left. \begin{aligned} \lceil 2\sqrt{4n+1} \rceil &\leq 2\chi(G) \leq 4n+2 \\ 4n+1 &\leq (\chi(G))^2 \leq \lfloor (\frac{4n+2}{2})^2 \rfloor \end{aligned} \right\} \quad (5.2)$$

Then it follows that

$$\lceil \sqrt{4n+1} \rceil \leq \chi(G) \leq 2n+1 . \quad \square$$

A characterisation for a s.c. perfect graph to be a chordal graph in terms of its chromatic number follows.

Theorem 5.3 : *Let G be a s.c. perfect graph. Then G is chordal if and only if*

$$(i) \ \chi(G) = 2n \text{ when } p=4n$$

$$(ii) \ \chi(G) = 2n+1 \text{ when } p=4n+1.$$

Proof: Let G be a chordal graph. By Theorem 2.20 $\omega(G) = 2n$ when $p=4n$ and $\omega(G) = 2n+1$ when $p=4n+1$. By the definition of a perfect graph $\chi(G) = \omega(G)$. So $\chi(G) = 2n$ when $p=4n$ and $\chi(G) = 2n+1$ when $p=4n+1$.

Let $\chi(G) = 2n$ when $p=4n$ and $\chi(G) = 2n+1$ when $p=4n+1$. By the definition of a perfect graph $\omega(G) = \chi(G)$. So $\omega(G) = 2n$ when $p=4n$ and $\omega(G) = 2n+1$ when $p=4n+1$. Then by Theorem 2.20 it follows that G is a chordal graph. \square

Next result gives bounds for the chromatic number of a s.c. perfect graph. The upper bounds given by this result is attained only for s.c. chordal graphs and every s.c. chordal graph attains the upper bound.

Theorem 5.4 : *Let G be a s.c. perfect graph. Then*

- (i) $\lceil \sqrt{4n} \rceil \leq \chi(G) \leq 2n$ when $p=4n$
- (ii) $\lceil \sqrt{4n+1} \rceil \leq \chi(G) \leq 2n+1$ when $p=4n+1$.

Moreover, the upper bounds given by (i) and (ii) are attained if and only if G is a chordal graph.

Proof: Follows from Theorem 5.2 and Theorem 5.3. \square

5.3 On the chromatic index of self-complementary chordal graphs

The following result was independently obtained by Behzad et. al. [28] and Vizing [227].

Theorem 5.5 (Behzad et. al. [28] and Vizing [227]) : *The complete graph K_{2n} where n is a positive integer is a Class 1 graph.*

A sufficient condition for a s.c. chordal graph with $4n$ vertices to be a Class 1 graph is given by the following Theorem.

Theorem 5.6 : *Let G be a s.c. chordal graph with $p=4n$. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ for all $1 \leq i \leq s$ be a star c.p. of G . Let H be the graph $\langle T^* \rangle$ where*

$T^* = [Even(\sigma^*), Odd(\sigma^*)]$. Then G is a Class 1 graph if both (i) and (ii) are satisfied.

(i) H is a Class 1 graph.

(ii) There exists a vertex v_{ij} of G such that $deg_G(v_{ij}) = \Delta(G)$ and $deg_H(v_{ij}) = \Delta(H)$.

Proof: Let H be a Class 1 graph. Let v_{ij} be a vertex of G such that $deg_G(v_{ij}) = \Delta(G)$ and $deg_H(v_{ij}) = \Delta(H)$. Let H' be the graph $\langle Even(\sigma^*) \rangle$. By Theorem 4.1 $v_{ij} \in Even(\sigma^*)$. The subgraph H is a Class 1 graph implies that the edges of H can be properly colored with the colors $a_1, a_2, \dots, a_{\Delta(H)}$. By Theorem 3.8 $H' \cong K_{2n}$ since $|Even(\sigma^*)| = 2n$. So $Odd(\sigma^*)$ is a stable set of G since $Odd(\sigma^*) = \sigma^*(Even(\sigma^*))$. By Theorem 5.5 the graph H' is a Class 1 graph since $H' \cong K_{2n}$. Hence the edges of H' can be properly colored with the colors $b_1, b_2, \dots, b_{2n-1}$. Every edge of G is either an edge of the graph H or an edge of the graph H' since $Odd(\sigma^*)$ is a stable set of G . So the edges of the graph G can be properly colored with the colors $a_1, a_2, \dots, a_{\Delta(H)}, b_1, b_2, \dots, b_{2n-1}$. The vertex $v_{ij} \in Even(\sigma^*)$ implies $deg_{H'}(v_{ij}) = 2n - 1$. Hence $\Delta(G) = deg_G(v_{ij}) = deg_H(v_{ij}) + deg_{H'}(v_{ij}) = \Delta(H) + 2n - 1$ since every edge incident with v_{ij} is either an edge of H or an edge of H' . Then it follows that G is a Class 1 graph since the edges of G can be properly colored with $\Delta(G)$ colors $a_1, a_2, \dots, a_{\Delta(H)}, b_1, b_2, \dots, b_{2n-1}$. \square

The following result is due to König [125].

Theorem 5.7 (König [125]) : Every bipartite graph is a Class 1 graph.

A sufficient condition for a s.c. chordal graph with $4n$ vertices to be a Class 1 graph in terms of its induced subgraph is given by the following result.

Theorem 5.8 : Let G be a s.c. chordal graph with $p=4n$. Let σ^* be a star c.p. of G . Then G is a Class 1 graph if $\langle T^* \rangle$ is a regular graph where $T^* = [Even(\sigma^*), Odd(\sigma^*)]$.

Proof: Let $\sigma^* = \sigma_1^* \sigma_2^* \dots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \dots v_{ip_i})$ for $1 \leq i \leq s$. Let H be the graph $\langle T^* \rangle$. The graph H is regular implies that $deg_H(v_{ij}) = \Delta(H)$ for all the vertices v_{ij} of G . So there exists a vertex v_{ij} of G such that $deg_G(v_{ij}) = \Delta(G)$ and $deg_H(v_{ij}) = \Delta(H)$ since $V(G) = V(H)$. We note that H is a bipartite graph since the sets $Even(\sigma^*)$ and $Odd(\sigma^*)$ partition $V(H)$ such that every edge of H is incident with a vertex of $Even(\sigma^*)$ and a vertex

of $Odd(\sigma^*)$. By Theorem 5.7 H is a Class 1 graph. Therefore by Theorem 5.6 the result follows. \square

Lemma 5.1 : *Let G be a s.c. graph with $4n$ vertices and a c.p. $\sigma = (1\ 2 \cdots 4n)$. Then $\langle T \rangle$ is a regular graph where $T = [Even(\sigma), Odd(\sigma)]$.*

Proof: Let H be the graph $\langle T \rangle$. By Lemma 2.2 $[j, \sigma^{2i-1}(j)] \in E(G)$ if and only if $[\sigma^{4n-(2i-1)}(j), \sigma^{4n}(j)] = [j, \sigma^{4n-(2i-1)}(j)] \notin E(G)$ for all $1 \leq i \leq n$ and $1 \leq j \leq 4n$. Hence $deg_H(i) = n$ for all the vertices i of G such that $1 \leq i \leq 4n$. \square

The following Theorem establishes the Class 1 property for a class of s.c. chordal graphs with $4n$ vertices.

Theorem 5.9 : *Let G be a s.c. chordal graph with $4n$ vertices. Let $\sigma^* = (1\ 2 \cdots 4n)$ be a star c.p. of G . Then G is a Class 1 graph.*

Proof: Follows from Theorem 5.8 and Lemma 5.1. \square

Alavi et. al. [4] and Vizing [227] independently obtained the following result.

Theorem 5.10 (Alavi et. al. [4] and Vizing [227]) : *Let G be a graph with p vertices. Then*

$$(i) \ 2\lfloor \frac{p+1}{2} \rfloor - 1 \leq \chi'(G) + \chi'(\bar{G}) \leq p + 2\lfloor \frac{p-2}{2} \rfloor$$

$$(ii) \ 0 \leq \chi'(G)\chi'(\bar{G}) \leq (p-1)(2\lfloor \frac{p}{2} \rfloor - 1).$$

The following result is on bounds for the chromatic index of a s.c. graph.

Theorem 5.11 : *Let G be a s.c. graph. Then*

$$(i) \ 2n \leq \chi'(G) \leq 4n - 1 \text{ when } p=4n$$

$$(ii) \ 2n + 1 \leq \chi'(G) \leq 4n - 1 \text{ when } p=4n+1.$$

Proof: Let $p=4n$. We note that $\chi'(G) = \chi'(\bar{G})$ since G is a s.c. graph. By Theorem 5.10 $\chi'(G)$ should satisfy the inequations given by (5.3).

$$\left. \begin{aligned} 2\lfloor \frac{4n+1}{2} \rfloor - 1 &\leq 2\chi'(G) \leq 4n + 2\lfloor \frac{4n-2}{2} \rfloor \\ 0 &\leq (\chi'(G))^2 \leq (4n-1)(2\lfloor \frac{4n}{2} \rfloor - 1) \end{aligned} \right\} \quad (5.3)$$

Hence

$$2n \leq \chi'(G) \leq 4n - 1.$$

(ii) Let $p=4n+1$. The graph G is a s.c. graph implies that $\chi'(G) = \chi'(\bar{G})$. By Theorem 5.10 $\chi'(G)$ should satisfy the inequations given by (5.4).

$$\left. \begin{aligned} 2\lfloor \frac{(4n+2)}{2} \rfloor - 1 &\leq 2\chi'(G) \leq 4n + 1 + 2\lfloor \frac{(4n-1)}{2} \rfloor \\ 0 &\leq (\chi'(G))^2 \leq 4n(2\lfloor \frac{(4n+1)}{2} \rfloor - 1) \end{aligned} \right\} \quad (5.4)$$

Therefore

$$2n + 1 \leq \chi'(G) \leq 4n - 1. \quad \square$$

Now we obtain bounds for the chromatic index of a s.c. chordal graph.

Theorem 5.12 : *Let G be a s.c. chordal graph. Then*

(i) $2n \leq \chi'(G) \leq 4n - 1$ when $p=4n$

(ii) $2n + 1 \leq \chi'(G) \leq 4n - 1$ when $p=4n+1$.

Proof: Follows from Theorem 5.11. \square

5.4 On the domination number of self-complementary chordal graphs

For a s.c. chordal graph with a star c.p. σ^* the set of all odd labelled vertices of σ^* is a kernel of the graph.

Theorem 5.13 : *Let G be a s.c. chordal graph. Let σ^* be a star c.p. of G . Then $Odd(\sigma^*)$ is a kernel of G .*

Proof: Case i : Let G be a s.c. chordal graph with $4n$ vertices. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ for $1 \leq i \leq s$. By Theorem 2.1 $p_i = 4n_i$ for some n_i where $1 \leq i \leq s$. Let $v_{ij} \in Even(\sigma^*)$. By Lemma 2.2 either $[v_{ij}, \sigma^*(v_{ij})] \in E(G)$ or $[\sigma^{*4n_i-1}(v_{ij}), \sigma^{*4n_i}(v_{ij})] = [v_{ij}, \sigma^{*4n_i-1}(v_{ij})] \in E(G)$. Also $v_{ij} \in Even(\sigma^*)$ implies that both the vertices $\sigma^*(v_{ij})$ and $\sigma^{*4n_i-1}(v_{ij})$ belong to $Odd(\sigma^*)$. Hence for all the vertices $v_{ij} \in Even(\sigma^*)$ there exists a

vertex $v_{lk} \in \text{Odd}(\sigma^*)$ such that $[v_{ij}, v_{lk}] \in E(G)$. By Theorem 3.8 $\langle \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . So the set $\text{Odd}(\sigma^*)$ is a stable set of G since $\text{Odd}(\sigma^*) = \sigma^*(\text{Even}(\sigma^*))$. Then it follows that $\text{Odd}(\sigma^*)$ is a kernel of G since the sets $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$ partition $V(G)$.

Case ii : Let G be a s.c. chordal graph with $4n+1$ vertices. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$ where $\sigma_1^* = (v_0)$ and $\sigma_i^* = (v_{i1} v_{i2} \cdots v_{ip_i})$ for $2 \leq i \leq s$. By Theorem 2.1 $p_i = 4n_i$ for some n_i where $2 \leq i \leq s$. Let $v_{ij} \in \text{Even}(\sigma^*)$. We note that v_{ij} is distinct from v_0 since $v_0 \in \text{Odd}(\sigma^*)$. By Lemma 2.2 either $[v_{ij}, \sigma^*(v_{ij})] \in E(G)$ or $[\sigma^{*4n_i-1}(v_{ij}), \sigma^{*4n_i}(v_{ij})] = [v_{ij}, \sigma^{*4n_i-1}(v_{ij})] \in E(G)$. Also $v_{ij} \in \text{Even}(\sigma^*)$ implies that both the vertices $\sigma^*(v_{ij})$ and $\sigma^{*4n_i-1}(v_{ij})$ belong to $\text{Odd}(\sigma^*)$. Hence for all the vertices $v_{ij} \in \text{Even}(\sigma^*)$ there exists a vertex $v_{lk} \in \text{Odd}(\sigma^*)$ such that $[v_{ij}, v_{lk}] \in E(G)$. By Theorem 3.13 $\langle \{v_0\} \cup \text{Even}(\sigma^*) \rangle$ is a complete subgraph of G . So the set $\text{Odd}(\sigma^*)$ is a stable set of G since $\text{Odd}(\sigma^*) = \sigma^*(\{v_0\} \cup \text{Even}(\sigma^*))$. It follows that $\text{Odd}(\sigma^*)$ is a kernel of G since the sets $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$ partition $V(G)$. \square

The following result gives an upper bound for the independent domination number of a s.c. chordal graph.

Theorem 5.14 : *Let G be a s.c. chordal graph. Then*

- (i) $\delta_i(G) \leq 2n$ when $p=4n$
- (ii) $\delta_i(G) \leq 2n + 1$ when $p=4n+1$.

Proof: Let σ^* be a star c.p. of G . By Theorem 5.13 $\text{Odd}(\sigma^*)$ is a kernel of G . Hence $\delta_i(G) \leq 2n$ and $\delta_i(G) \leq 2n + 1$ accordingly as $p=4n$ and $p=4n+1$ respectively since $|\text{Odd}(\sigma^*)| = 2n$ when $p=4n$ and $|\text{Odd}(\sigma^*)| = 2n + 1$ when $p=4n+1$. \square

Lemma 5.2 : *For any graph G , $\delta_0(G) \leq \delta_i(G)$.*

Proof: Follows from the definition of $\delta_0(G)$ and $\delta_i(G)$. \square

The following result gives an upper bound for the domination number of a s.c. chordal graph.

Theorem 5.15 : *Let G be a s.c. chordal graph. Then*

- (i) $\delta_0(G) \leq 2n$ when $p=4n$

(ii) $\delta_0(G) \leq 2n + 1$ when $p=4n+1$.

Proof: Follows from Theorem 5.14 and Lemma 5.2. \square

5.5 On the spectrum of self-complementary chordal graphs

No two non-isomorphic s.c. chordal graph with p vertices where $p < 12$ have the same spectrum.

Theorem 5.16 : *No two s.c. chordal graph with 4 vertices are cospectral.*

Proof: Vacuously true since G has only one non-isomorphic s.c. chordal graph with 4 vertices. \square

Theorem 5.17 : *No two s.c. chordal graph with 5 vertices are cospectral.*

Proof: Vacuously true since G has only one non-isomorphic s.c. chordal graph with 5 vertices. \square

Theorem 5.18 : *No two s.c. chordal graph with 8 vertices are cospectral.*

Proof: Adjacency matrices of the three non-isomorphic s.c. chordal graphs with 8 vertices and their characteristic polynomial are given below.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $16\lambda^2 + 40\lambda^3 + 21\lambda^4 - 16\lambda^5 - 14\lambda^6 + \lambda^8$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $1 - 4\lambda - 6\lambda^2 + 20\lambda^3 + 11\lambda^4 - 20\lambda^5 - 14\lambda^6 + \lambda^8$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $-16\lambda - 12\lambda^2 + 32\lambda^3 + 25\lambda^4 - 16\lambda^5 - 14\lambda^6 + \lambda^8$.

We note that none of the above polynomials can be obtained from any one of the other two polynomials by multiplying by a real number. Hence no pair of non-isomorphic s.c. chordal graphs with 8 vertices have the same spectrum. \square

Theorem 5.19 : *No two s.c. chordal graph with 9 vertices are cospectral.*

Proof: Adjacency matrices of the three non-isomorphic s.c. chordal graphs with 9 vertices and their characteristic polynomial are given below.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $32\lambda^3 + 68\lambda^4 + 25\lambda^5 - 28\lambda^6 - 18\lambda^7 + \lambda^9$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $5\lambda - 16\lambda^2 - 6\lambda^3 + 40\lambda^4 + 11\lambda^5 - 32\lambda^6 - 18\lambda^7 + \lambda^9$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial of the above matrix is $-32\lambda^2 - 12\lambda^3 + 60\lambda^4 + 29\lambda^5 - 28\lambda^6 - 18\lambda^7 + \lambda^9$.

We note that none of the above polynomials can be obtained from any one of the other two polynomials by multiplying by a real number. Hence no pair of non-isomorphic s.c. chordal graphs with 9 vertices have the same spectrum. \square

Cospectral s.c. chordal graphs with 12 vertices exists.

Theorem 5.20 : *There exists cospectral s.c. chordal graphs with 12 vertices.*

Proof: Consider the two graphs with 12 vertices shown in Figure 5.1 and Figure 5.2. They are s.c. graphs since $(1\ 11\ 2\ 12)(6\ 9\ 5\ 8\ 4\ 7\ 3\ 10)$ and $(1\ 12\ 2\ 11)(6\ 7\ 5\ 8)(3\ 10\ 4\ 9)$ are c.p.'s of the graphs in Figure 5.1 and Figure 5.2 respectively. By Theorem 2.20 both these graphs are chordal since their clique number is 6. These two graphs are non-isomorphic since the graph in Figure 5.1 has a vertex of degree 1 whereas the graph in Figure 5.2 does not have a vertex of degree 1.

The adjacency matrix and the characteristic polynomial of the s.c. chordal graph shown in Figure 5.1 follows.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of the above matrix is $-16\lambda + 20\lambda^2 + 112\lambda^3 - 91\lambda^4 - 256\lambda^5 + 83\lambda^6 + 248\lambda^7 + 20\lambda^8 - 88\lambda^9 - 33\lambda^{10} + \lambda^{12}$.

The adjacency matrix and the characteristic polynomial of the s.c. chordal graph shown in Figure 5.2 follows.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the above matrix is $-16\lambda + 20\lambda^2 + 112\lambda^3 - 91\lambda^4 - 256\lambda^5 + 83\lambda^6 + 248\lambda^7 + 20\lambda^8 - 88\lambda^9 - 33\lambda^{10} + \lambda^{12}$.

We note that the characteristic polynomial of the s.c. chordal graphs with 12 vertices shown in Figure 5.1 and Figure 5.2 are the same. Therefore they have the same spectrum. \square

The following result is a consolidation of the above results.

Theorem 5.21 : *The smallest positive integer for which there exist cospectral s.c. chordal graphs is 12.*

Proof: Follows from Theorem5.16, Theorem5.17, Theorem5.18, Theorem5.19 and Theorem5.20. \square

Remark: In the above results characteristic polynomials of the adjacency matrices of s.c. chordal graphs were computed using the package 'Mathematica'. For more details about this package we refer to [232].

The following result gives bounds for the maximum eigenvalue of a graph.

Theorem 5.22 (Schwenk and Wilson [193]) : *Let G be a connected graph. Then either (i) or (ii) holds.*

$$(i) \frac{2q}{p} < \lambda_{\max}(G) < \Delta(G)$$

$$(ii) \frac{2q}{p} = \lambda_{\max}(G) = \Delta(G), \text{ } G \text{ is a regular graph and } (1, 1, \dots, 1) \text{ is an eigen vector of } G.$$

Lemma 5.3 : *Let G be a s.c. chordal graph. Then G is not regular.*

Proof: Follows from Theorem4.1 and Theorem4.2. \square

Lemma 5.4 : *Let G be a s.c. graph. Then G is connected.*

Proof: For every graph either the graph or its complement is connected. Hence G is connected since G is a s.c. graph. \square

Lemma 5.5 : *Let G be a s.c. graph with p vertices. Then $q = \frac{p(p-1)}{4}$.*

Proof: The graph K_p has $\frac{p(p-1)}{2}$ edges. The graph G is s.c. implies it has exactly half the number of edges of K_p . Hence G has $\frac{p(p-1)}{4}$ edges. \square

Bounds for the maximum eigen value of a s.c. chordal graph are given by the following result.

Theorem 5.23 : *Let G be a s.c. chordal graph. Then*

- (i) $\frac{(4n-1)}{2} < \lambda_{max}(G) < \Delta(G)$ when $p=4n$
- (ii) $2n < \lambda_{max}(G) < \Delta(G)$ when $p=4n+1$.

Proof: Let G be a s.c. chordal graph. By Lemma5.3 and Lemma5.4 G is connected and not regular. Then by Theorem5.22 and Lemma5.5 the result follows. \square

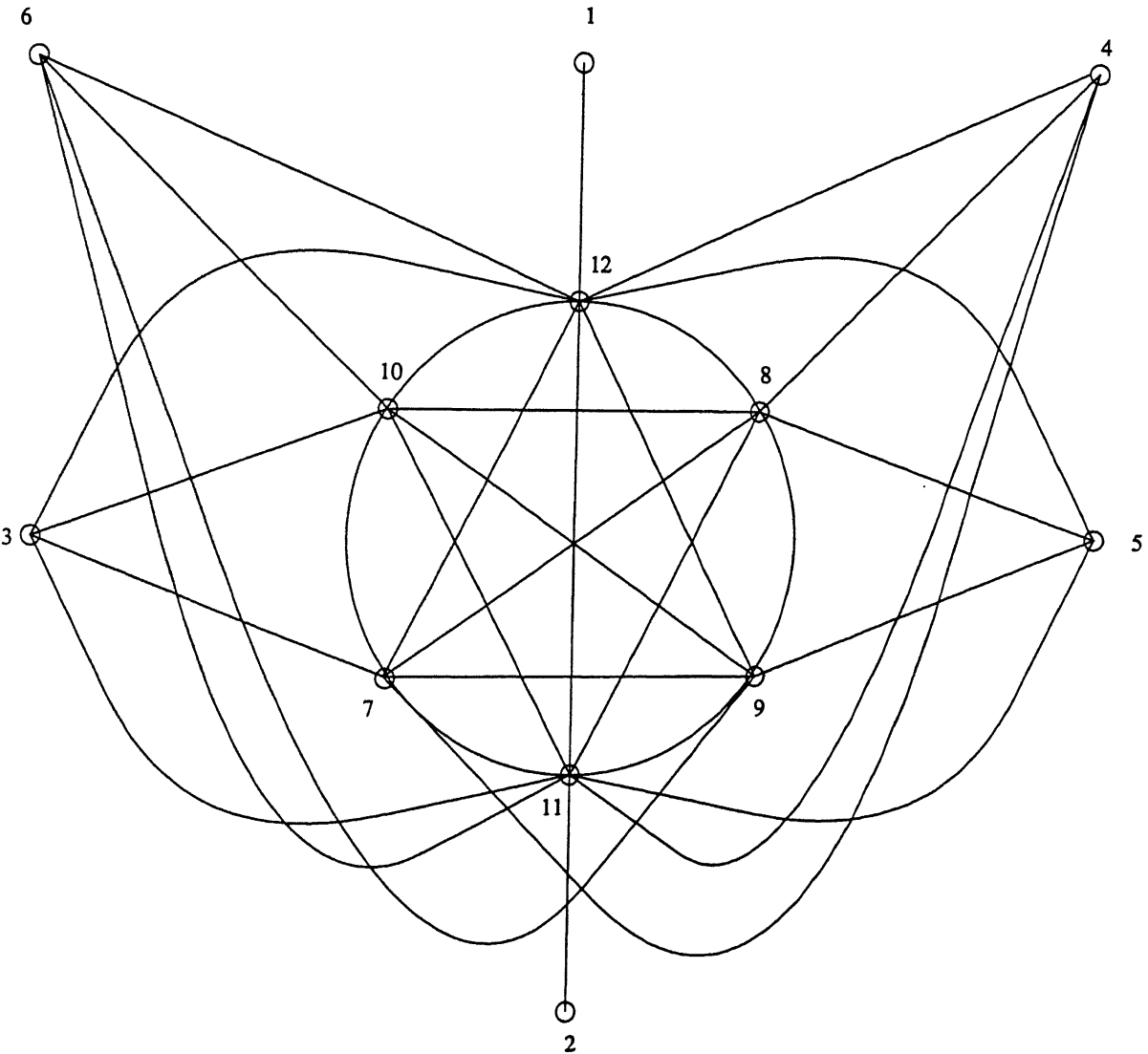


Figure 5.1

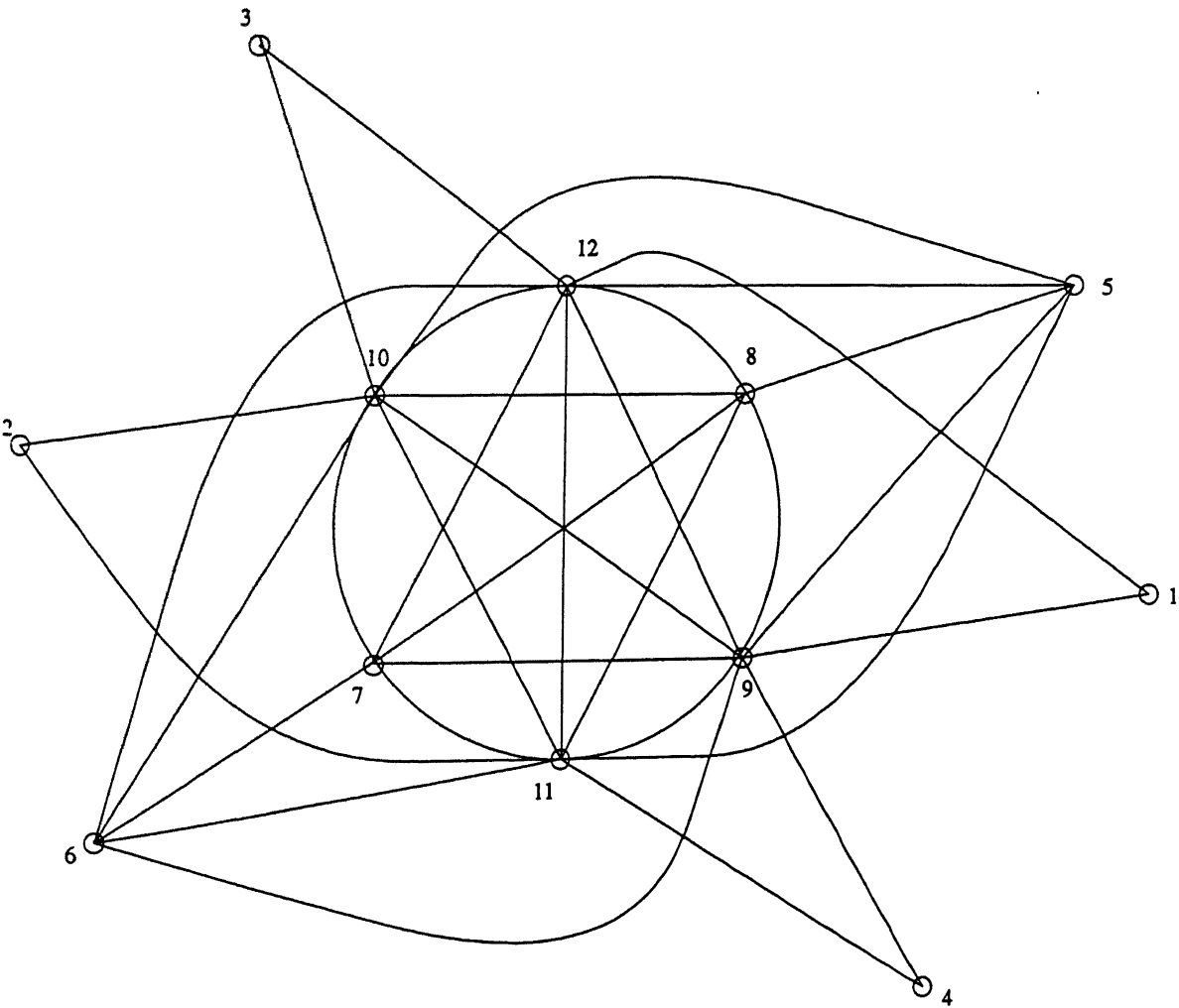


Figure 5.2

Chapter 6

Conclusion

In this Chapter we pose some research problems on s.c. chordal graphs.

In Chapter4 we proved that the isomorphism of s.c. chordal graphs is polynomially equivalent to the isomorphism of s.c. chordal graphs with $4n$ vertices and the isomorphism of s.c. chordal graphs with $4n+1$ vertices. We conjecture the following on the algorithmic complexity of the isomorphism of s.c. chordal graphs.

Conjecture 6.1 : *The isomorphism of s.c. chordal graphs is polynomially equivalent to the general graph isomorphism (the isomorphism of graphs).*

By Theorem4.11, Conjecture6.1 could be equivalently posed as follows.

Conjecture 6.2 : *The isomorphism of s.c. chordal graphs with $4n$ vertices (or $4n+1$ vertices) is polynomially equivalent to the general graph isomorphism (the isomorphism of graphs).*

In Chapter 4 we proved that recognising whether a given s.c. graph is chordal or not takes linear time. However, the problem of recognising whether a given graph is s.c. chordal or not (the recognition problem of s.c. chordal graphs) seems to be algorithmically as hard as that of the general graph isomorphism. So we pose the following Conjecture.

Conjecture 6.3 : *The recognition problem of s.c. chordal graphs is polynomially equivalent to the general graph isomorphism (the isomorphism of graphs).*

From Theorem 4.16 it follows that the number of non-isomorphic s.c. chordal graphs with $4n$ vertices is equal to the number of non-isomorphic s.c. chordal graphs with $4n+1$ vertices. However the problem of counting the number of non-isomorphic s.c. chordal graphs with $4n$ vertices or $4n+1$ vertices for each positive integer n seems to be quite difficult. Hence we pose this as an open problem.

Problem 6.1 : *Count the number of non-isomorphic s.c. chordal graphs with $4n$ vertices for each positive integer n .*

Equivalently the above problem could be posed as follows.

Problem 6.2 : *Count the number of non-isomorphic s.c. chordal graphs with $4n+1$ vertices for each positive integer n .*

Bibliography

- [1] A.F.Aho, J.E.Hopcroft and J.D.Ullman, 'The Design and Analysis of Computer Algorithms' (Addison-Wesley, Reading, MA, 1974).
- [2] J.Akiyama, G.Exoo and F.Harary, 'A graph and its complement with specified properties. V. The self-complementary index', *Mathematika* 27 (1980), 64-68.
- [3] J.Akiyama and F.Harary, 'A graph and its complement with specified properties. IV. Counting self-complementary blocks', *J. Graph Theory* 5 (1981), 103-107.
- [4] Y.Alavi and M.Behzad, 'Complementary graphs and edge chromatic numbers', *SIAM J. Appl. Math.* 20 (1971), 161-163.
- [5] R.Alter, 'A characterisation of self-complementary graphs of order 8', *Portug. Math.* 34 (1975), 157-161.
- [6] R.Ando, 'Rao's conjecture on self-complementary graphs with k-factors', *J. Graph Theory* 9 (1985), 119-121.
- [7] L.W.Beineke and R.Pippert, 'Properties and characterisation of k-trees', *Mathematica* 18 (1971), 141-151.
- [8] A.Benhocine and A.P.Wojda, 'On self-complementation', *J. Graph Theory* 9 (1985), 335-341.
- [9] C.Benzaken, P.L.Hammer and D.De Warra, 'Split graphs of Dilworth number 2', *Dis. Math.* 55 (1985), 123-127.

- [10] C.Benzaken, Y.Crama, P.Duchet, P.L.Hammer and F.Maffray, 'More characterisations of triangulated graphs', *J. Graph Theory* 14(4) (1990), 413-422.
- [11] S.Benzer, 'On the topology of the genetic fine structure', *Proc. Nat. Acad. Sci., U.S.A.* 45 (1959), 1607-1620.
- [12] C.Berge, 'Les problemes de colorations en theorie des graphes', *Publ. Inst. Stat. Univ. Paris* 9 (1960), 123-160.
- [13] C.Berge, 'Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind', *Wiss. Z. Martin-Luther Univ., Halle-Wittenberg Math-Natur., Reihe* (1961), 114-115.
- [14] C.Berge, 'Sur une conjecture relative au probleme des codes optimaux', *comm. 13e assemblee generale de l'URSI, Tokyo* (1962).
- [15] C.Berge, 'Une application de la theorie des graphes a un probleme de codage', *Automata Theory, Internat. School Phys. Ravello 1964* (1966), 95-134.
- [16] C.Berge, 'Some classes of perfect graphs', *Graph Theory and Theoretical Physics*, Academic Press, New York (1967), 155-165.
- [17] C.Berge, 'The rank of a family of sets and some applications to graph theory', in: 'Recent advances in Combinatorics' (W.T.Tutte ed.), Academic Press, New York, 1969.
- [18] C.Berge, 'Une propriete des graphes K-stables-critique', *Combinat. Struct. Appl.*, Calgary 1969 (1970), 7-11.
- [19] C.Berge, 'Introduction a la-theorie des hypergraphes', *Lecture Notes, Universite de Montreal*, Summer (1971).
- [20] C.Berge, 'Graphs and Hypergraphs' (North-Holland, Amsterdam, 1973).
- [21] C.Berge, 'Perfect Graphs', in: 'Studies in Graph Theory', Part 1 (D.R.Fulkerson ed.), M.A.A. Studies in Mathematics, Vol.2, Math. Assoc. Amer., Washington, D.C. (1975), pp 1-22.

- [22] C.Berge, 'Short note about the history of the perfect graph conjecture' (mimeographed) (1976).
- [23] C.Berge, 'New classes of perfect graphs', *Dis. App. Math.* 15 (1986), 145-154.
- [24] C.Berge and V.Chvatal, 'Topics on Perfect Graphs', *Ann. Dis. Math.* 21 (1984).
- [25] C.Berge and M.Las Vergnas, 'Sur un theoreme dee type König pour hypergraphes', in: 'Ann. NewYork Acad. Sci. 1975' (1976), pp 32-40.
- [26] A.A.Bertossi, 'Finding Hamiltonian circuits in proper interval graphs', *Info. Proc. Lett.* 17 (1983), 97-101.
- [27] A.A.Bertossi and M.A.Bonuccelli, 'Hamiltonian circuits in interval graph generalisations', *Info. Proc. Lett.* 23 (1986), 195-200.
- [28] M.Behzad, G.Chartrand and J.K.Cooper, 'The colour numbers of complete graphs', *J. London Math. Soc.* 42 (1967), 226-228.
- [29] R.G.Bland, H.C.Huang and L.E.Trotter Jr., 'Graphical properties related to minimal imperfection', *Dis. Math.* 27 (1979), 11-22.
- [30] A.Blokhuis and T.Kloks, 'On the equivalence covering number of split graphs', *Inform. Proc. Lett.* 54(5) (1995), 301-304.
- [31] K.S.Booth, 'Dominating sets in chordal graphs', Rep. No. CS-80-34, Comp. Sci. Dept., Univ. of Waterloo, Waterloo, Ontario, 1980.
- [32] K.S.Booth and G.S.Leuker, 'Testing for consecutive ones property, interval graphs and graph planarity using PQ-tree algorithm', *J. Comp. Sys. Sci.* 13 (1976), 335-379.
- [33] K.C.Booth and C.J.Colbourn, 'Problems polynomially equivalent to graph isomorphism', Tech. Rep. CS-77/04, Department of Computer Science, University of Waterloo (1979).
- [34] J.Bosak, 'Decomposition of Graphs' (Kluwer, Netherlands, 1990), pp. 175-184.

- [35] N.G.de Bruijn, 'Generalisation of Polya's fundamental theorem in enumeration combinatorial analysis', *Indag. Math.* 21 (1959), 59-69.
- [36] N.G.de Bruijn, 'Polya's theory of counting', in: 'Applied Combinatorial Mathematics' (E.F.Beckenbach ed.), John-Wiley, NewYork, 1964.
- [37] M.A.Buckingham and M.C.Golumbic, 'Partitionable graphs, circle graphs and the Berge strong perfect graph conjecture', *Dis. Math.* 44 (1983), 187-194.
- [38] P.Buneman, 'The recovery of trees from measures of dissimilarity', *Mathematics in Archaeological and Historical Sciences*, Edinburg Univ. Press, Edinburg (1972), 387-395.
- [39] P.Buneman, 'A characterisation of Rigid Circuit Graphs', *Dis. Math.* 9 (1974), 205-212.
- [40] R.E.Burkard and P.L.Hammer, 'A note on hamiltonian split graphs (A note)', *J. Comb. Theory B* 28 (2) (1980), 245-248.
- [41] P.Camion, 'Hamiltonian chains in self-complementary graphs', in: *Colloque sur la theorie des graphes*, Paris, 1974 (P.P.Gillis and S.Huyberechts eds.), *Cahiers du Centre d' Etudes de Recherche Operationelle* 17 (2-4) (1975), Inst. Stat. U.L.B. Bruxelles, 1975, 173-183.
- [42] O.M.Carducci, 'The stong perfect graph conjecture holds for diamonded odd cycle-free graph', *Dis. Math.* 110 (1992), 17-34.
- [43] N.Chandrasekaran and S.S.Iyengar, 'NC Algorithms for recognising chordal graphs and k-trees', *IEEE Trans. Comp.* 37 (1988), 1178-1183.
- [44] R.Chandrasekaran and A.Tamir, 'Polynomially bounded algorithms for locating p-centers on a tree', *Math. Programming* 22 (1982), 304-315.
- [45] C.Y.Chao and E.G.Whitehead Jr., 'Chromaticity of self-complementary graphs', *Arch. Math.* 32 (1979), 295-304.
- [46] Zh.A.Chernyak, 'Edge degree sequences in self-complementary graphs', *Mat. Zametki* 34 (1983), 297-308.

- [47] G.L.Chia and C.K.Lim, 'A class of self-complementary vertex transitive digraphs', J. Graph Theory 10 (1985), 241-249.
- [48] V.Chvatal, 'Star - cutsets and perfect graphs', J. Comb. Theory B 39 (1985), 189-199.
- [49] V.Chavtal, 'On the P_4 -structure of perfect graphs III : Partner Decomposition (Note)', J. Comb. Theory B 43 (1987).
- [50] V.Chvatal and P.L.Hammer, 'Set packing and threshold graphs', Univ. Waterloo Res. Report, CORR 73-21, 1973.
- [51] V.Chvatal, P.Erdos and Z.Hedrlin, 'Ramsey's theorem and self-complementary graphs', Dis. Math. 3 (1972), 301-304.
- [52] V.Chvatal, R.L.Graham, A.F.Perold and S.H.Whitesides, 'Combinatorial designs related to the strong perfect graph conjecture', Dis. Math. 26 (1979), 83-92.
- [53] V.Chavtal, W.J.Lenhart and N.Sbihi, 'Two colorings that decompose perfect graphs', J. Comb. Theory B 49(1) (1990), 1-9.
- [54] C.R.J.Clapham, 'Triangles in self-complementary graphs', J. Comb. Theory B 15 (1973), 74-76.
- [55] C.R.J.Clapham, 'Hamiltonian arcs in self-complementary graphs', Dis. Math. 8 (1974), 251-255.
- [56] C.R.J.Clapham, 'Hamiltonian arcs in infinite self-complementary graphs', Dis. Math. 13 (1975), 307-314.
- [57] C.R.J.Clapham, 'Potentially self-complementary degree sequences', J. Comb. Theory B 20 (1976), 75-79.
- [58] C.R.J.Clapham, 'Piecing together paths in self-complementary graphs', in: Proc. 5th British Combinatorial Conference, Aberdeen, 1975 (C.St.J.A.NashWilliams and J.Sheehan eds.), J. Congressus Numerantium 15, Utilitas Math., Winnipeg, 1976.

- [59] C.R.J.Clapham, 'A class of self-complementary graphs and lower bounds of some Ramsey numbers', J. Graph Theory 3 (1979), 287-289.
- [60] C.R.J.Clapham, 'Graphs self-complementary in $K_n - e$ ', Dis. Math. 8 (1990), 229-235.
- [61] C.R.J.Clapham and D.J.Kleitman, 'The degree sequences of self-complementary graphs', J. Comb. Theory B 20 (1976), 67-74.
- [62] E.J.Cockayne and S.T.Heidetniemi, 'Towards a theory of domination in graphs', Networks 7 (1977), 247-261.
- [63] C.J.Colbourn, 'A bibliography of the graph isomorphism problem', Tech. Rep. 123/78, Department of Computer Science, University of Toronto (1978).
- [64] C.J.Colbourn, 'On testing isomorphism of permutation graphs', Networks 11 (1981), 13-21.
- [65] M.J.Colbourn and C.J.Colbourn, 'Graph Isomorphism of self-complementary graphs', SIGACT News 10(1) (1978), 25-29.
- [66] C.J.Colbourn and M.J.Colbourn, 'Isomorphism involving self-complementary graphs and tournaments', in: Proc. 8th Manitoba Conference on Numerical Mathematics and Computing, Univ. Manitoba, Winnipeg, 1978 (D.McCarthy and H.C.Williams eds.), Congressus Numerantium 22, Utilitas Math., Winnipeg (1979), 153-164.
- [67] C.J.Colbourn and L.K.Stewart, 'Dominating cycles in series- parallel graphs', Ars Combinatoria 19A (1985), 107-112.
- [68] D.G.Corneil, 'Graph Isomorphism', Technical Report No. 18, Department of Computer Science, University of Toronto (1970).
- [69] D.G.Corneil, 'Recent results on the graph isomorphism problem', Proceedings of 8th Manitoba Conference on Numerical Mathematics and Computing (1978).
- [70] D.G.Corneil, 'Families of graphs complete for the Strong-Perfect Graph Conjecture', J. Comb. Theory 10(1) (1986), 37-40.

- [71] D.M.Cvetkovic, 'The generating function for variations with restrictions and paths of the graph and self-complementary graphs', Univ. Beograd Publ. Elektrotehn. Fak., Ser. Math. Fiz. (320-328) (1970), 27-34.
- [72] D.M.Cvetkovic, M.Doob, I.Gutman and A.Torgasev, 'Recent results in graph spectra', Annal. Dis. Math. 36, North-Holland, 1988.
- [73] Das Prabir, 'Characterisation of potentially self-complementary, self-converse degree-pair sequences for digraphs', in: 'Combinatorics and Graph Theory', Proc. 2nd Symp. Indian Stat. Inst., Calcutta, 1980 (S.B.Rao ed.), Lecture Notes in Math. 885, Springer Verlag (1981).
- [74] Das Prabir, 'Integer-pair sequences with self-complementary realizations', Dis. Math. 45 (1983), 189-198.
- [75] G.A.Dirac, 'On rigid circuit graph', Abh. Math. Sem. Univ. Hamburg 25 (1961), 71-76.
- [76] A.Edenbrandt, 'Chordal graph recognition is in NC', Info. Proc. Lett. 24 (1987), 239-241.
- [77] R.Fagin, 'Degrees of acyclicity for hypergraphs and relational database schemes', J.A.C.M. 30 (1983), 514-550.
- [78] I.A.Faradzhev, 'A complete list of self-complementary graphs with 12 or less vertices', in: 'Algorithmic Investigations in Combinatorics' (I.A.Faradzhev ed.), Nauka, Moscow, 1978, 69-75.
- [79] M.Farber, 'Independent domination in chordal graphs', Oper. Res. Lett. 4 (1982), 134-138.
- [80] M.Farber, 'Characterisations of strongly chordal graphs', Dis. Math. 43 (1983), 173-187.
- [81] S.Fiorini, 'A bibliographic survey of edge-colourings', J. Graph Theory 2 (1978), 93-106.

-
- [82] S.Fiorini and R.J.Wilson, 'Edge-colourings of graphs' (Pitman, London, 1977).
 - [83] S.Fiorini and R.J.Wilson, 'Edge-colourings of graphs', in: 'Selected Topics in Graph Theory' (L.W.Beineke and R.J.Wilson eds.), Academic Press, London, 1978, 103-126.
 - [84] D.Foata, 'Enumerating k-trees', *Dis. Math.* 1 (1972), 181-186.
 - [85] S.Foldes and P.L.Hammer, 'Split Graphs', in: 'Proc. 8th South-Eastern Conference on Combinatorics, Graph Theory and Computing' (1977), 311-315.
 - [86] S.Foldes and P.L.Hammer, 'Split graphs having Dilworth number two', *Can. J. Math.* 29 (3) (1977), 666-672.
 - [87] S.Foldes and P.L.Hammer, 'On a class of matroid producing graphs', in: 'Combinatorics' (A.Hajnal and V.T.Sos eds.), Vol.I, North-Holland, 1978.
 - [88] D.Froncek, A.Rosa and J.Siran, 'The existence of self-complementary circulant graphs', *Euro. J. Math.* 17 (1996), 625-628.
 - [89] D.R.Fulkerson and O.A.Gross, 'Incidence matrices and interval graphs', *Pacific J. Math.* 15 (1965), 835-855.
 - [90] J.Fulman, 'A note on the characterisation of domination perfect graphs', *J. Graph Theory* 17 (1) (1993), 47-51.
 - [91] R.Garey and D.S.Johnson, 'Computers and Intractability : A Guide to the Theory of NP-Completeness', (W.H. Freeman, New York, 1979).
 - [92] G.Gati, 'Further Annotated Bibliography on the Isomorphism Disease', *J. Graph Theory* 3(2) (1979), 95-109.
 - [93] F.Gavril, 'Algorithms for minimum coloring, maximum cliques, minimum covering by cliques and maximum independent set in a chordal graph', *SIAM J. Comput.* 1 (1972), 180-187.
 - [94] F.Gavril, 'An algorithm for testing chordality of graphs', *Inf. Proc. Lett.* 3 (1974), 110-112.

- [95] F.Gavril, 'The intersection graphs of subtrees are exactly the chordal graphs', J. Comb. Theory B 16 (1974), 47-56.
- [96] F.Gavril, 'A recognition algorithm for the intersection graphs of directed paths in directed trees', Dis. Math. 13 (1975), 237-249.
- [97] F.Gavril, 'A recognition algorithm for the intersection graphs of paths in a tree', Dis. Math. 23 (1978), 211-217.
- [98] R.A.Gibbs, 'Self-Complementary Graphs', J. Comb. Theory B 16 (1974), 106-123.
- [99] P.C.Gilmore and A.T.Hoffmann, 'A characterisation of comparability graphs and of interval graphs', Canad. J. Math. 16 (1964), 539-548.
- [100] M.C.Golumbic, 'Trivially perfect graphs - A Note', Dis. Math. 24 (1978), 50-71.
- [101] M.C.Golumbic, 'Algorithmic Graph Theory and Perfect Graphs' (Academic Press, New York, 1980).
- [102] M.C.Golumbic (ed.), 'Interval graphs and related topics', Dis. Math. 55 (1985).
- [103] M.C.Golumbic and R.E.Jamison, 'The edge intersection graphs of paths in a tree', J. Comb. Theory B 38 (1985), 8-22.
- [104] C.M.Grinstead, 'The perfect graph conjecture for toroidal graphs', J. Comb. Theory B 30 (1981), 70-74.
- [105] A.Grotschel, L.Lovasz and A.Schrijver, 'Polynomial algorithms for perfect graphs', Ann. Dis. Math. 21 (1984), 325-356.
- [106] A.Hajnál, and J.Surányi, 'Über die Auflösung von Graphen in vollständige Teilgraphen', Ann. Univ. Sci. Budapest Etövös. Sect. Math. 1 (1958) 113-121.
- [107] G.Hajös, 'Über eine Art von Graphen', Intern. Math. Nachr. 11 , Problem 65.
- [108] P.L.Hammer and F.Maffray, 'More characterisations of triangulated graphs', J. Graph Theory 14 (4) (1990), 413-422.

- [109] P.L.Hammer, and B.Simone, 'The Splittance of a Graph', *Combinatorica* 1 (1981), 375-384.
- [110] P.L.Hammer, T.Ibaraki and B.Simone, 'Degree sequences of threshold graphs', Univ. Waterloo Res. Report, CORR 78-10, 1978.
- [111] F.Harary, 'Unsolved problems in the enumeration of graphs', *Publ. Math. Inst. Hung. Acad. Sci. A5* (1960), 63-95.
- [112] F.Harary, 'Graph Theory' (Addison-Wesley, Reading, MA, 1969).
- [113] F.Harary and E.M.Palmer, 'Graphical Enumeration' (Academic Press, New York , 1973).
- [114] N.Hartsfield, 'On regular self-complementary graphs', *J. Graph Theory* 11 (1987), 537-538.
- [115] M.Hegde, R.C.Read and M.R.Sridharan, 'The enumeration of transitive self-complementary digraphs', *Dis. Math.* 47 (1983), 109-112.
- [116] S.T.Heidetniemi and R.Laskar, 'Recent results and open problems in domination theory', *Applications of Dis. Math.*, SIAM, Philadelphia, PA, 1988, 205-218.
- [117] S.T.Heidetniemi and R.C.Laskar (eds.), 'Topics on domination', *Dis. Math.* 86 (1-3) (Special Volume) (1990).
- [118] C.T.Hoang, 'On the Sibling-Structure of Perfect Graphs', *J. Comb. Theory B* 49(2) (1990). 282-286.
- [119] W.L.Hsu, 'Decomposition of Perfect Graphs', *J. Comb. Theory B* 43 (1987), 70-94.
- [120] L.Hubert, 'Some applications of graph theory and related non-metric techniques to problems of approximate seriation. The case of symmetric proximity measures', *British J. Math. Stat. Psych.* 27 (1974), 133-153.

- [121] R.E.Jamison and R.Laskar, 'Elimination orderings of chordal graphs', in: Proc. of the seminar on combinatorics and applications (K.S.Vijayan et. al. (eds.)) I.S.I. Calcutta (1982), 192-200.
- [122] T.R.Jensen and B.Toft, 'Graph Coloring Problems' (John Wiley and Sons, 1995).
- [123] D.S.Johnson, 'The NP-Completeness Column : An Ongoing Guide', J. Algorithms 6 (3)(1985), 434-451.
- [124] P.N.Klein, 'Efficient parallel algorithms for chordal graphs', in: Proc. 29th Annual IEEE Symp. on Foundations of Comp. Sci. (1988), 150-161.
- [125] D.König, 'Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre', Math. Ann. 77 (1916), 453-465.
- [126] N.Korte and R.H.Mohoring, 'An incremental linear time algorithm for recognising interval graphs', SIAM J. Comput. 18 (1989), 68-81.
- [127] M.Kropar and R.C.Read, 'On the construction of self-complementary graphs on 12 nodes', J. Graph Theory 3 (1979), 111-125.
- [128] R.Laskar and D.R.Shier, 'On chordal graphs', Congres. Numer. 29 (1980), 579-588.
- [129] V.B.Le, 'Perfect k-line graphs and k-total graphs', J. Graph Theory 17 (1993), 63-73.
- [130] C.G.Lekkerker and J.C.H.Boland, 'Representations of a finite graph by a set of intervals on the real line', Fund. Math. 51 (1962), 45-64.
- [131] G.S.Leuker, 'Structured breadth first search and chordal graphs', Prin. Univ. Tech. Rep. TR-158, 1974.
- [132] L.Lovasz, 'A characterisation of perfect graphs', J. Comb. Theory B(12) (1972), 95-98.
- [133] L.Lovasz, 'Normal hypergraphs and the perfect graph conjecture', Dis. Math. 2 (1972), 253-267.
- [134] L.Lovasz, 'On the Shannon capacity of a graph', IEEE Trans. Inf. Theory 25 (1979), 1-7.

- [135] L.Lovasz, 'Perfect graphs', in: 'Selected Topics in Graph Theory – 2' (Wilson and Beineke (eds.)), Academic Press, 1983, 55-87.
- [136] A.Lubev, 'Short chorded and perfect graphs', J. Comb. Theory B 51(1) (1991), 24-33.
- [137] R.D.Luce, 'Periodic extensive measurement', Compositio Math. 23 (1971), 189-198.
- [138] F.Maffaray and M.Preissmann, 'Perfect Graphs with no P_5 and no K_5 ', Graphs and Combinatorics 10 (1994), 179-184.
- [139] F.Maffray and M.Preissmann, 'Split-neighborhood graphs and the strong perfect graph conjecture', J. Comb. Theory B 63(2) (1995), 294-309.
- [140] R.Mathon, 'On self-complementary strongly regular graphs', Dis. Math. 69 (1988), 263-281.
- [141] H.Meyneil, 'On the perfect graph conjecture', Dis. Math. 16 (1976), 339-342.
- [142] C.L.Monma and V.K.Wei, 'Intersection graphs of paths in a tree', J. Comb. Theory B 41 (1986), 141-181.
- [143] P.A.Morris, 'On self-complementary graphs and digraphs', in: Proc. 5th South-Eastern Conference on Combinatorics, Graph Theory and Computing, Florida Atlantic Univ., Boca Raton, 1974 (F.Hoffmann et. al., eds.), Congressus Numerantium 10, Utilitas Math., Winnipeg, (1974), 583-590.
- [144] J.Noar, M.Noar and A.Schaffer, 'Fast parallel algorithms for chordal graphs', SIAM J. Comput. 18 (1989), 327-349.
- [145] E.A.Nordhaus and J.W.Gaddum, 'On complementary graphs', Amer. Math. Monthly 63 (1956), 175-177.
- [146] S.Olariu, 'The strong perfect graph conjecture for pan-free graphs', J. Comb. Theory B 21 (1976), 212-223.
- [147] S.Olariu, 'On the Strong Perfect Graph Conjecture', J. Graph Theory 12(2) (1988), 169-176.

- [148] E.Olaru and E.Mandrescu, 'S-Strongly perfect cartesian products of graphs', J. Graph Theory 16(4) (1992), 297-303.
- [149] E.M.Palmer, 'Asymptotic fomulas for the number of self-complementary graphs and digraphs', Mathematika 17 (1970), 85-90.
- [150] C.Papadimitriou and M.Yanakakis, 'Scheduling interval ordered tasks', SIAM J. Comp. 8 (1979), 405-409.
- [151] K.R.Parthasarathy, 'Basic Graph Theory' (Tata McGraw-Hill, New Delhi, 1994).
- [152] K.R.Parthasarathy and G.Ravindra, 'The stong perfect graph conjecture is true for $K_{1,3}$ -free graphs', J. Comb. Theory B 21 (1976), 212-223.
- [153] K.R.Parthasarathy and G.Ravindra, 'The validity of the strong perfect graph conjecture for $(K_4 - e)$ -free graphs', J. Comb. Theory B 26 (1979), 98-100.
- [154] K.R.Parthasarathy and M.R.Sridharan, 'Enumeration of self-complementary graphs . and digraphs', J. Math. Phy. Sci. 3 (1969), 410-414.
- [155] A.Pneuli, A.Lempel an S.Even, 'Transitive orientation of graphs and identification of permutation graphs', J. Assoc. Comput. Mach. 19 (1971), 400-410.
- [156] G.Polya, 'Kombinatorische Anzahlbestimmungen fur Gruppen, Graphen and chemische Verbindungen', Acta Math. 68 (1937), 145-254.
- [157] M.Preissmann, 'Locally Perfect Graphs', J. Comb. Theory B 50(1) (1990), 22-40.
- [158] A.Prokurowski, 'k-trees : Representations and distances', Tech. Rep., CIS-TR-80-5, Univ. Oregon, OR, 1980.
- [159] A.Prokurowski, 'Separating subgraphs in k-trees, cables and caterpillars', Dis. Math. 49 (1984), 275-285.
- [160] B.Radhakrishnan Nair and A.Vijayakumar, 'About triangles in a graph and its complement', Dis. Math. 131 (1994), 205-210.

- [161] S.B.Rao, 'Graph and its complement', *Ind. Nat. Sci. Acad.* A41 (3) (1975), 297-304.
- [162] S.B.Rao, 'Characterisation of self-complementary graphs with 2-factors', *Dis. Math.* 17 (1977), 225-233.
- [163] S.B.Rao, 'Cycles in self-complementary graphs', *J. Comb. Theory B* 22 (1977), 1-9.
- [164] S.B.Rao, 'Explored, semi-explored and unexplored territories in the structure theory of self-complementary graphs and digraphs', in: *Proc. Symp. on Graph Theory*, Indian Stat. Inst., Calcutta, 1976 (A.R.Rao ed.), ISI Lecture Notes 4, Macmillan of India, New Delhi, 1979.
- [165] S.B.Rao, 'Solution of the Hamiltonian problem for self-complementary graphs', *J. Comb. Theory B* (1979), 13-41.
- [166] S.B.Rao, 'The number of open chains of length three and the parity of the number of open chains of length k in self-complementary graphs', *Dis. Math.* 28 (1979), 291-301.
- [167] S.B.Rao, 'The range of the number of triangles in self-complementary graphs of given order', in: *Proc. Symp. on Graph Theory*, Indian Stat. Inst., Calcutta, 1976 (A.R.Rao ed.), ISI Lecture Notes 4, Macmillan of India, New Delhi, 1979.
- [168] S.B.Rao, 'On regular and strongly-regular self-complementary graphs', *Dis. Math.* 54 (1985), 73-82.
- [169] S.B.Rao and G.Ravindra, 'A characterisation of perfect graphs', *J. Math. and Phys. Sci.* 11 (1977), 25-26.
- [170] G.Ravindra, 'On Berge's conjecture concerning perfect graphs', *Proc. Indian Nat. Sci. Acad.* 41A, 294-296.
- [171] R.C.Read, 'On the number of self-complementary graphs and digraphs', *J. London Math. Soc.* 38 (1963), 99-104.
- [172] R.C.Read, 'Some applications of a theorem of de Bruijn', in: 'Graph Theory and Theoretical Physics' (F.Harary ed.), Academic Press, London, 1967, 273-280.

- [173] R.C.Read, 'A survey of graph generation techniques', in: 'Combinatorial Mathematics VIII', Proc. 8th Australian Conf. on Combinatorial Mathematics, Deakin Univ., Geelong, 1980 (K.L.McAvaney ed.), Lecture Notes in Mathematics 884, Springer-Verlag (1981), 77-89.
- [174] R.C.Read and D.G.Corneil, 'The Graph Isomorphism Disease', J. Graph Theory 1 (1977), 339-363.
- [175] J.H.Redfield, 'The Theory of Group-Reduced Distributions', Amer. J. Math. 49 (1927), 433-455.
- [176] B.Reed, 'A Semi-Strong Perfect Graph Theorem', J. Comb. Theory B 43 (1987), 70-94.
- [177] P.L.Renz, 'Intersection representations of graphs by arcs', Pacific J. Math. 34 (1970), 501-510.
- [178] G.Ringel, 'Selbstkomplementäre Graphen', Arch Math. (Basel) 14 (1963), 354-358.
- [179] F.S.Roberts, 'Discrete Mathematical Models with Applications to Social Biological and Environmental Problems' (Prentice-Hall, New Jersey, 1976).
- [180] R.W.Robinson, 'Asymptotic number of self-converse oriented graphs', in: 'Combinatorial Mathematics', Proc. Int. Conf. on Combinatorial Theory, Australian Nat. Univ., Canberra, 1977 (D.A.Holton and J.Seberry eds.), Lectures Notes in Math. 686, Springer-Verlag, 1978, 255-266.
- [181] D.Rose, 'Triangulated graphs and the elimination process', J. Math. Anal. App. 32 (1970), 597-609.
- [182] D.Rose, 'A graph theoretic study of the numerical solution of sparse positive definite systems of linear equations', in: 'Graph theory and Computing' (R.C.Read ed.), Academic Press, New York (1972), 183-217.
- [183] D.Rose, 'Single characterisations of k -trees', Dis. Math. 7 (1974), 317-322.
- [184] D.J.Rose and R.E.Tarjan, 'Algorithmic aspects of vertex elimination', Proc. 7th Annual ACM Symp. Theo. Comput. Sci. (1975), 245-254.

- [185] D.J.Rose and R.E.Tarjan, 'Algorithmic aspects of vertex elimination of directed graphs', SIAM J. App. Math. 34 (1978), 176-197.
- [186] D.J.Rose, R.E.Tarjan and G.Leuker, 'Algorithmic aspects of vertex elimination on graphs', SIAM J. Comput. 5 (1976), 266-283.
- [187] I.G.Rosenberg, 'Regular and strongly regular self-complementary graphs', in: Theory and Practice of Combinatorics (A.Rosa et. al., eds.), North-Holland Mathematics Study 60, Annals of Dis. Math. 12, North-Holland, 1982, 223-238.
- [188] S.Ruiz, 'On a family of self-complementary graphs', in: Combinatorics '79, Part II, Proc. Colloq. on Univ. Montreal, Montreal, 1979 (M.Deza and I.G.Rosenberg eds.), Annals of Discrete Mathematics 8-9, North-Holland (1980), 267-268.
- [189] S.Ruiz, 'On strongly regular self-complementary graphs', J. Graph Theory 5 (1981), 213-215.
- [190] H.Sachs, 'Über selbstkomplementäre Graphen', Publ. Math. Debrecen 9(1962), 270-288.
- [191] H.Sachs, 'Remarks on the construction of cyclic self-complementary directed graphs', (German), Wissen Z. Tech. Hochsch., Ilmenau 11 (1965), 161-162.
- [192] A.J.Schwenk, 'The number of color cyclic factorizations of a graph', (Enumeration and Design, Academic Press, Toronto, 1984), 297-311.
- [193] A.J.Schwenk and R.J.Wilson, 'Eigenvalues of graphs', in: 'Selected Topics in Graph Theory' (L.W.Beineke and R.J.Wilson eds.), Academic Press, New York, 1978, 307-336.
- [194] D.Seinche, 'On a Property of the Class of n -colorable Graphs', J. Comb. Theory B(16)(1974), 191-193.
- [195] C.E.Shannon, 'Zero-error capacity of a noisy channel', Comp. Info. Theory, IRE Trans. 3 (1956), 3-15.

- [196] Y.Shibata, 'On the tree representation of chordal graphs', *J. Graph Theory* 12 (1988), 421-428.
- [197] D.R.Shier, 'Some aspects of perfect elimination orderings in chordal graphs', *Dis. Appl. Math.* 7 (1984), 325-331.
- [198] C.deSimone and A.Galluccio, 'New Classes of Berge Perfect Graphs', *Discrete Math.* 131 (1994), 67-79.
- [199] P.Sreenevasa Kumar, 'Algorithmic and structural results on chordal graphs', Ph.D Thesis, I.I.Sc. Bangalore, 1990.
- [200] P.Sreenevasa Kumar and C.E.Veni Madhavan, 'A new class of separators and planarity of chordal graphs', *Lecture Notes in Comp.Sci.* 405 (1989), 30-43.
- [201] R.Sridhar and S.S.Iyengar, 'Fast parallel algorithms for recognizing strongly chordal, ptolemaic and block graphs', *Int. Conf. on Parallel Processing* (1990), III-141-III-144.
- [202] M.R.Sridharan, 'Self-complementary and self-converse oriented graphs', *Nederl. Akad. Wetensch. Proc. A73, Indag. Math.* 32 (1970), 441-447.
- [203] M.R.Sridharan, 'Note on an asymptotic formula for a class of digraphs', *Can. Math. Bull.* 21 (1978), 377-381.
- [204] M.R.Sridharan and K.Balaji, 'On self-complementary chordal graphs', *National Academy Science Letters* 20 (1997).
- [205] M.R.Sridharan and K.Balaji, 'Characterisation of self-complementary chordal graphs', accepted for publication in *Discrete Mathematics*.
- [206] M.R.Sridharan and K.Balaji, 'On construction of self-complementary chordal graphs', submitted for publication.
- [207] M.R.Sridharan and K.Balaji, 'On isomorphism of self-complementary chordal graphs', submitted for publication.

- [208] M.R.Sridharan and O.T.George, 'On the property of strong perfect graph conjecture (A note)', Dis. Math. 41 (1982), 101-104.
- [209] M.R.Sridharan and K.R.Parthasarathy, 'Isographs and oriented isographs', J. Comb. Theory B 13 (1972), 99-111.
- [210] F.W.Stahl, 'Circular genetic maps', J. Cell. Physiol. Suppl. 70 (1967), 1-12.
- [211] K.E.Stoffers, 'Scheduling of traffic lights - a new approach', Transport. Res. 2 (1968), 199-234.
- [212] L.Sun, 'Two classes of Perfect Graphs', J. Comb. Theory B 53 (1991), 273-292.
- [213] D.A.Suprunenko, 'Self-complementary graphs', Cybernetics 21 (1985), 559-567.
- [214] M.M.Syslo, 'Triangulated edge intersection graphs of paths in a tree', Dis. Math. 55 (1985), 217-220.
- [215] R.E.Tarjan, 'Maximum cardinality search and chordal graphs', Unpublished Lecture Notes, Stanford University, 1976.
- [216] R.E.Tarjan and M.Yanakakis, 'Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs and selectively reduce acyclic hypergraphs', SIAM J. Comput. 13 (1984), 225-331.
- [217] B.Toft, 'Graph colouring : A survey of some problems and results', Second Danish-Polish Math. Prog. Seminar, Compenhagen, 1979.
- [218] W.T.J.Trotter, 'A note on triangulated graphs', Notices Amer. Math. Soc. 18 1045 (A) (1971).
- [219] L.E.Trotter Jr., 'On line perfect graphs', Math. Prog. 12 (1977), 255-259.
- [220] A.C.Tucker, 'The strong perfect graph conjecture and an application to a municipal routing problem', in: 'Graph Theory and Applications', Proc. Conference of Western Michingan Univ., Kalamazoo, Lect. Notes in Math. 303 (1972), Springer-Verlag, 297-303.

- [221] A.C.Tucker, 'The strong perfect graph conjecture for planar graphs', *Canad. J. Math.* 25 (1973), 103-114.
- [222] A.C.Tucker, 'Critical perfect graphs and perfect 3-chromatic graphs', *J. Comb. Theory B* 23 (1977), 143-149.
- [223] A.C.Tucker, 'Circular arc graphs - new uses and a new algorithm', in: 'Theory and applications of graphs', *Lecture Notes in Math.* 642, Springer Verlag (1978), 580-589.
- [224] A.C.Tucker, 'Uniquely colorable perfect graphs', *Dis. Math.* 44 (1983), 187-194.
- [225] C.V.Venkatachalam, 'Self-complementary graphs on eight points', *Math. Education* 10 (2) (1976), A43-A44.
- [226] V.G.Vizing, 'On an estimate of the chromatic class of a p-graph' (Russian), *Diskret. Analiz.* 3 (1964), 25-30.
- [227] V.G.Vizing, 'The chromatic class of a multigraph', *Cybernetics* 1 (1965) (3), 32-41.
- [228] W.D.Wallis and J.Wu, 'Squares, clique graphs and chordality', *J. Graph Theory* 20(1) (1995), 37-45.
- [229] J.R.Walter, 'Representation of chordal graphs as subtrees of a tree', *J. Graph Theory* 2 (1978), 265-267.
- [230] D.deWarra, 'On line perfect graphs', *Math. Prog.* 15 (1978), 236-238.
- [231] K.White, M.Farber and W.Pulleyblank, 'Steiner trees, connected domination and strongly chordal graphs', *Networks* 15 (1985), 109-124.
- [232] S.Wolfram, 'Mathematica : A system for doing mathematics by computer' (Addison-Wesley, Reading, MA, 1988).
- [233] J.Wu, 'Chordal graph and bipartite representation of graphs', Ph.D. Dissertation, Department of Mathematics, Shandong University, Junan, Shandong, People's Republic of China (1987).

-
- [234] M.Yanakakis and F.Gavril, 'The maximum k-colorable subgraph problem for chordal graphs', Inf. Proc. Lett. 24 (1987), 133-137.
- [235] B.Zelinka, 'Self-complementary vertex transitive undirected graphs', Maths. Slovaca 29 (1979), 91-95.
- [236] H.Zhang, 'Self-complementary symmetric graphs', J. Graph Theory 16 (1) (1992), 1-5.

A 31082

A 131082
Date Sup

This book is to be returned on the date last stamped.

[illegible]

A131082

A131082

TH

MATH/1997/P

B1820